WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 541 Probability and Mathematical Statistics II

August 2025

Note: Please show clear, logical reasoning for each solution. Even if you cannot solve a problem completely, clearly outline your approach. For example, if you run out of time before finishing a calculation, demonstrate the steps you would take to complete it and, if applicable, how you would use the result in subsequent steps.

Problem 1 (20 points)**.** Let the pdf of a logistic location distribution be

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2} \ , \quad -\infty < x < \infty, -\infty < \theta < \infty.$$

Does this distribution family have a monotone likelihood ratio (MLR)? Why or why not?

Problem 2 (20 points)**.** Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \ \theta > 0.$$

a. Find a sufficient statistic for θ.

b. Show that the statistic you found in (a) is complete.

c. Let $g(\theta) = \frac{1}{\theta}$. Find the UMVUE of $g(\theta)$.

Problem 3 (20 points)**.** Let observations (X_i, Y_i) be independent and identically distributed for $i = 1, ..., n$, where $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$, $\epsilon_i \sim N(0,1)$, X_i is a random variable with a known distribution, and ϵ_i and X_i are independent.

(a) Identify the lowest possible variance for an unbiased estimator of β_1.

(b) Give an estimator of β_1 that achieves the variance identified in part (a).

Problem 4 (20 points)**.** Let $X_1, X_2, \ldots, X_n$ be iid random variables with $N(0, \sigma^2)$. Consider testing the hypotheses:

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{against} \quad H_1 : \sigma^2 = \sigma_1^2,$$

where $\sigma_0^2 \neq \sigma_1^2$. What is the most powerful unbiased test at significance level α?

Problem 5 (20 points)**.** Consider the following probability density function (pdf) of X,

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda}{2\mu^2}\frac{(x-\mu)^2}{x}\right\}, \quad \mu, \lambda, x > 0,$$

where we can write $X \sim \text{IG}(\mu, \lambda)$. Let $X_1, \ldots, X_n \mid \mu \overset{ind}{\sim} \text{IG}(\mu, \lambda)$, where λ is assumed known.

a. Find the MLE, $\hat{\mu}$, of μ.

b. Find the distribution of $\hat{\mu}$.

c. Find $\mathrm{E}(\hat{\mu})$.

Hint: The moment generating function (MGF) of $X \sim \mathrm{IG}(\mu, \lambda)$ is

$$M_X(t) = \exp\left\{\frac{\lambda}{\mu}\left(1 - \sqrt{1 - \frac{2\mu^2 t}{\lambda}}\right)\right\}, t < \frac{\lambda}{2\mu^2}.$$

Problem 6 (20 points)**.** In clinical trials in medicine, patients randomized to the treatment condition may refuse to take the treatment. To model this scenario, say samples were taken from n patients randomized to the treatment condition: $\{Y_1^1, X_1\}, \{Y_2^1, X_2\}, \ldots, \{Y_n^1, X_n\}$, where Y^1 are outcomes (for instance, survival or quality of life) and $X_i = 1$ if subject i took the treatment and $X_i = 0$ if not. Say a separate i.i.d. sample of outcomes was taken from n patients assigned to the control condition: $Y_1^0, Y_2^0, \ldots, Y_n^0$. For simplicity, assume these two samples were independently drawn from separate infinite superpopulations.

The "instrumental variables" estimator for the causal effect of taking the medicine is

$$\hat{\beta}_{IV} = \frac{\overline{Y^1} - \overline{Y^0}}{\overline{X}}$$

What is the asymptotic distribution of $\hat{\beta}_{IV}$ as $n \to \infty$? Express the distribution in terms of n and the following quantities: $\mathbb{E}[Y^1] = \mu_1$, $\mathbb{E}[Y^0] = \mu_0$, $Var(Y^1) = Var(Y^0) = \sigma_Y^2$, $\mathbb{E}[X] = p > 0$, and $Cov(Y^1, X) = \sigma_{XY}$.