GCE: 503, Analysis and measure theory

August 2017

No documents, no calculators allowed

Write your name on each page you turn in

Exercise 1:

Let h_n be a sequence of non-negative, borel measurable functions on the interval (0,1) such that $h_n \to 0$ in $L^1((0,1))$.

(i). Show $\sqrt{h_n} \to 0$ in $L^1((0,1))$.

(ii). Give an example to show that h_n^2 need not converge to zero in $L^1((0,1))$.

(iii). If g_n is in $L^1(\mathbb{R})$ such that $|g_n|^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$, and g_n converges to zero in $L^1(\mathbb{R})$ as n tends to infinity, does $|g_n|^{\frac{1}{2}}$ converge to zero in $L^1(\mathbb{R})$?

Exercise 2:

Let f be in $L^\infty((0,1))$. Show that $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Exercise 3:

Let a_n be a sequence in $[0,1]$ such that the set $S = \{a_n : n = 1, 2, ...\}$ is dense in $[0,1]$. Set

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - a_n|^{-\frac{1}{2}}}{n^2}$$

(i). Show that f is in $L^1([0,1])$.

(ii). Is f in $L^2([0,1])$?

(ii). Is there a continuous function

$$g : [0,1] \setminus S \to \mathbb{R}$$

such that $f = g$ almost everywhere?

Exercise 4:

Let $\mathcal{R}$ be the set of all rectangles $(a_1, b_1) \times (a_2, b_2)$ in $\mathbb{R}^2$ such that a_1, b_1, a_2, b_2 are rational numbers.

(i). Let V be an open set in $\mathbb{R}^2$. Show that

$$V = \bigcup_{R \in \mathcal{R}, R \subset V} R.$$

(ii). Recall that the Borel sets of $\mathbb{R}^2$ are the sets in the smallest sigma algebra of $\mathbb{R}^2$ containing all open sets. Show that the smallest sigma algebra of $\mathbb{R}^2$ containing $\mathcal{R}$ is equal to the set of Borel sets of $\mathbb{R}^2$.