GCE: 503, Analysis and measure theory
May 2016

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Exercise 1:

A real-valued function f is *increasing* on a closed interval $[a, b] \subset \mathbb{R}$ if and only if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.

(i). Using the definition of *measurable*, show that f is measurable on $[a, b]$.

(ii). Show that f is continuous, except perhaps a countable number of points.

Exercise 2:

If f is Lebesgue integrable on $\mathbb{R}$, define

$$F(x) = \int_0^x f d\mu$$

where $\mu(E)$ is the Lebesgue measure of any measurable set $E \subset \mathbb{R}$. Show that

(i). F is continuous.

(ii). if $\mu(E) = 0$, then $\mu(F(E)) = 0$.

Exercise 3:

Let f be in $L^1(\mathbb{R})$ such that $f \geq 0$ almost everywhere and $\int_{\mathbb{R}} f = 1$. Set $f_n(x) = nf(nx)$. Let g be in $L^\infty(\mathbb{R})$.

(i). Let x_0 be in $\mathbb{R}$. Assume that g is continuous at x_0. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x_0 - y)g(y)dy = g(x_0)$$

(ii). If g is uniformly continuous, is this limit uniform in x_0?

(iii). If h is in $L^1(\mathbb{R})$ show that the function in x

$$\int_{\mathbb{R}} f_n(x - y)h(y)dy$$

converges to h in $L^1(\mathbb{R})$.