Exercise 1:
A real-valued function $f$ is increasing on a closed interval $[a, b] \subset \mathbb{R}$ if and only if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.

(i). Using the definition of measurable, show that $f$ is measurable on $[a, b]$.
(ii). Show that $f$ is continuous, except perhaps a countable number of points.

Exercise 2:
If $f$ is Lebesgue integrable on $\mathbb{R}$, define

$$F(x) = \int_0^x f d\mu$$

where $\mu(E)$ is the Lebesgue measure of any measurable set $E \subset \mathbb{R}$. Show that

(i). $F$ is continuous.
(ii). if $\mu(E) = 0$, then $\mu(F(E)) = 0$.

Exercise 3:
Let $f$ be in $L^1(\mathbb{R})$ such that $f \geq 0$ almost everywhere and $\int_\mathbb{R} f = 1$. Set $f_n(x) = nf(nx)$.

Let $g$ be in $L^\infty(\mathbb{R})$.

(i). Let $x_0$ be in $\mathbb{R}$. Assume that $g$ is continuous at $x_0$. Show that

$$\lim_{n \to \infty} \int_\mathbb{R} f_n(x_0 - y)g(y)dy = g(x_0)$$

(ii). If $g$ is uniformly continuous, is this limit uniform in $x_0$?
(iii). If $h$ is in $L^1(\mathbb{R})$ show that the function in $x$

$$\int_\mathbb{R} f_n(x - y)h(y)dy$$

converges to $h$ in $L^1(\mathbb{R})$. 