Worcester Polytechnic Institute
Department of Mathematical Sciences

May 2015

General Comprehensive Examination

Linear Algebra

Print Name: ______________________ Sign: ______________________

Unless otherwise specified, all matrices are assumed to have complex entries.

Problem 1: Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

(a) Determine if among A,B,C there is a pair of similar matrices.

(b) For the matrix C find the transformation matrix M and its inverse such that $J = M^{-1}CM$ is the Jordan canonical form of C.

Problem 2: In the matrix

$$A = \begin{pmatrix} 0 & a & 0 \\ a & t & a \\ 0 & a & 0 \end{pmatrix}$$

can you specify a constant a such that A has an eigenvalue $\lambda = \lambda(t)$ such that $d\lambda/dt = 2$ for $t = 1$?

Problem 3: Prove that, if A is nilpotent, then $A + I$ is invertible.

Problem 4: Let $\mathbf{P}_3(t)$ denote the vector space of polynomials with real coefficients of degree three or less in t. Consider the linear transformation $\mathsf{T} : \mathbf{P}_3(t) \to \mathbf{P}_3(t)$ given by

$$a + b\,t + c\,t^2 + d\,t^3 \quad \mapsto \quad (b+c) + (a+d)t + (a+d)t^2 + (b+c)t^3.$$

With respect to the standard basis $\{1, t, t^2, t^3\}$ find matrices representing orthogonal projections from $\mathbf{P}_3(t)$ onto each of the eigenspaces of T.

Problem 5: Consider the vector space of polynomials on $[-1, 1]$ with real coefficients with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)(1-t^2)\,dt\,.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis, with respect to this inner product, for the subspace spanned by $\left\{\frac{1}{2}\sqrt{3},\ \frac{1}{2}\sqrt{15}t,\ t^2\right\}$.

(b) Is this inner product non-degenerate? Is it positive definite? Justify your answer.

Problem 6(a): Let A and B be invertible $n \times n$ matrices. Prove that, if the matrix

$$M = \begin{pmatrix} A & B \\ B^{-1} & A^{-1} \end{pmatrix}$$

also has rank n, then A and B commute.

(b) Now show how to diagonalize M in terms of given diagonalizations of A and B.