Exercise 1:
Let $K$ be $\mathbb{R}$ or $\mathbb{C}$ and $a_0, \ldots, a_{n-1}$ be in $K$. Let $C$ be the $n$ by $n$ matrix
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & & \vdots & \ddots \\
0 & 0 & \cdots & 1 & -a_{n-1}
\end{pmatrix}
\]
Show that the characteristic polynomial of $C$ and the minimal polynomial of $C$ are both equal to $P(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$.

Exercise 2:
Let $A$ and $B$ be two invertible $n$ by $n$ matrices. Let $M$ be the matrix
\[
M = \begin{pmatrix} A & B \\ B^{-1} & A^{-1} \end{pmatrix}.
\]
Assume that $M$ has rank $n$. Show that $A$ and $B$ commute.

Exercise 3:
Let $A$ be an $n$ by $n$ invertible matrix with entries in $K$. Suppose that $u$ and $v$ are two vectors in $K^n$ such that $1 + v^T A^{-1} u \neq 0$. Show that $A + uv^T$ is invertible.
\textbf{Hint}: It suffices to prove that the inverse of $A + uv^T$ is
\[
A^{-1} - m A^{-1} uv^T A^{-1}
\]
where $m$ is an adequate scalar.

Exercise 4:
Let $A = (a_{ij})$ be a symmetric $n$ by $n$ matrix with real entries. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. Show that
\[
\sum_{1 \leq i, j \leq n} a_{ij}^2 = \sum_{i=1}^n \lambda_i^2
\]