Category I problems

problem I.1

Let $A$ be the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Compute
(a) the rank of $A$,
(b) the trace of $A$,
(c) the determinant of $A$,
(d) the characteristic polynomial of $A$,
(e) all the eigenvalues of $A$,
(f) find corresponding eigenvectors,
(g) Compute $(2I - A)^3 - 2(2I - A)$

Does the answer to (g) surprise you? Can you formulate a relation of $A$ to its characteristic polynomial?

(h) Is $A$ diagonalizable? If so, find an orthogonal matrix $Q$ such that $A = Q\Lambda Q^T$

(i) Compute $A^T A$ and $AA^T$ and their eigenvalues and eigenvectors.
(j) Compute $e^A$ and comment (without computing it) on the behavior of the solution to $\frac{d\tilde{u}}{ds} = A\tilde{u}, \tilde{u}(0) = \tilde{u}_0$.

problem 1.2

Consider the linear map $T : P_3 \to P_2$ defined by differentiation, i.e., by $T(p) = p' \in P_2$ for $p \in P_3$. Find the matrix representation of $T$ with respect to the bases $\{1 + x, 1 - x, x + x^2, x^2 - x^3\}$ for $P_3$ and $\{1, x, x^2\}$ for $P_2$.

problem 1.3

Compute the matrix of transformation of coordinates (back and forth) from the canonical basis in $\mathbb{R}^2$ to the basis

$$B = \left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$$

(these vectors' coordinates are with respect to the canonical basis).

problem 1.4

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}.$$

Find the least square solution of $Ax = b$.

problem 1.5

The set of all real $n \times n$ matrices, denoted $\mathbb{R}^{n \times n}$, is a vector space under the usual operations of matrix addition and scalar multiplication. Consider $S \equiv \{ A \in \mathbb{R}^{n \times n} : A^T = -A \}$, the set of all skew-symmetric matrices in $\mathbb{R}^{n \times n}$.

(a) Show that $S$ is a subspace of $\mathbb{R}^{n \times n}$.

(b) Show that $P : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ defined by $P(A) = \frac{1}{2} (A + A^T)$ is the projection of $\mathbb{R}^{n \times n}$ onto $S$ that is orthogonal with respect to the Frobenius inner product.

Category II problems
**problem II.1**
Prove that if $A$ is a non-singular $n \times n$ matrix, then there exists a polynomial $f(t)$ such that $Af(A) = f(A)A = I$.

**problem II.2**
Prove that any square $n \times n$ matrix $A$ can be obtained as a limit of matrices $A_I \to A$ that have $n$ distinct eigenvalues.