STUDENT NAME:

Linear Algebra Graduate Comprehensive Exam, January 2017
Worcester Polytechnic Institute

Work out at least 6 of the following problems
Write down detailed proofs of every statement you make.
No Books. No Notes. No calculators.

1. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

(i) Find the transformation matrix M, and its inverse, such that $A = MJM^{-1}$ is the Jordon canonical form of A.

2. Consider the linear vector space $\mathbb{C}^n$ with inner product $\langle x, y\rangle = \sum_{i=1}^n \bar{x}_i y_i$ for $x, y \in \mathbb{C}^n$. (i) Prove the Cauchy-Schwarz inequality.

3. Let A be a $n \times m$ real matrix. (i) Prove that $Ax = b$ has at least one solution if and only if $b \in \mathcal{N}(A^T)^\perp$.

4. The Tchebyshev polynomials (with real coefficients) $T_0(x), T_1(x), T_2(x), \ldots, T_n(x)$ on the interval $[-1, 1]$ can be characterized by:

(1) For each $0 \le i \le n$, the polynomial $T_i(x)$ has degree i

(2) For each $0 \le i \le n$, $T_i(1) = 1$

(3) For each $0 \le i \le n$ and $0 \le j \le n$ such that $i \ne j$, T_i and T_j are orthogonal with respect to the following weighted inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}dx$$

(i) Find $T_0(x), T_1(x)$ and $T_2(x)$. Hint: Use $x = -\cos(\theta)$

5. (i) Find the Euclidean orthogonal projection of the vector w on the subspace $S = \text{Span}\{v_1, v_2\}$ described by

$$w = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

6. Let A be $n \times m$ real matrix. (i) If the eigenvalues of A are all distinct, then A is diagonalizable. (ii) If in addition $A = A^T$, then the eigenvectors are orthogonal to each other.

7. Let A and B be two $n \times n$ matrices. (ii) Prove that $\text{Rank}(AB) \le \min\{\text{Rank}(A), \text{Rank}(B)\}$