STUDENT NAME:

Linear Algebra Graduate Comprehensive Exam, May 2016
Worcester Polytechnic Institute

Work out at least 5 of the following problems
Write down detailed proofs of every statement you make.
No Books. No Notes. No calculators.

1. Find an example of two matrices A, B such that
$$\cos(A+B) \neq \cos A \cos B - \sin A \sin B$$

2. Let A be a real $n \times n$ matrix. Define $\langle x, y \rangle := \sum_{i,j=1}^{n} a_{ij} x_i y_j$. Find necessary and sufficient conditions on A for this operation to be a inner product on $\mathbb{R}^3$.

3. Consider the vector space of polynomials with real coefficients and with inner product
$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)(1-t^2)dt.$$
Apply the Gram-Schmidt process to find an orthonormal basis, with respect to this inner product, for the subspace generated by $\{\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{2}t, t^2\}$.

4. Prove that the function
$$A \to \max_{|v|=1} |Av|$$
defines a norm (called the operator norm) on the space of square matrices with real coefficients.

5. Let A be an $n \times 2n$ matrix with entries $a_{i,i} = a_{i,2n-i+1} = 1$ for $1 \leq i \leq n$, $a_{i,i-1} = a_{i,2n-i+2} = k$, for $2 \leq i \leq n$ (k is a positive constant), and 0 otherwise.
(i). Compute the rank of A.
(ii). Find a basis for the row space of A, the column space of A and the nullspace of A.
(iii). Does the system $Ax = b$ have a solution if b is a vector all of whose entries are 1?
If there is a solution, find all solutions. If there is no solution, find the least squares solution.
(iv). Does the system $A^T y = c$ have a solution, where c is a vector all of whose entries are 1? If there is a solution, find all solutions. If there is no solution, find the least squares solution.

6. Let A be a $n \times n$ matrix. Prove that there exists a $n \times n$ matrix B such that $AB = 0$ and $rank(A) + rank(B) = n$.