1. Find an example of two matrices \( A, B \) such that
\[
\cos(A + B) \neq \cos A \cos B - \sin A \sin B
\]

2. Let \( A \) be a real \( n \times n \) matrix. Define \( \langle x, y \rangle := \sum_{i,j=1}^{n} a_{ij} x_i y_j \). Find necessary and sufficient conditions on \( A \) for this operation to be a inner product on \( \mathbb{R}^3 \).

3. Consider the vector space of polynomials with real coefficients and with inner product
\[
\langle f, g \rangle := \int_{-1}^{1} f(t) g(t)(1 - t^2)dt.
\]
Apply the Gram-Schmidt process to find an orthonormal basis, with respect to this inner product, for the subspace generated by \( \{ \sqrt{3}, \sqrt{15}, t, t^2 \} \).

4. Prove that the function
\[
A \mapsto \max_{|v|=1} |Av|
\]
defines a norm (called the operator norm) on the space of square matrices with real coefficients.

5. Let \( A \) be an \( n \times 2n \) matrix with entries \( a_{i,i} = a_{i,2n-i+1} = 1 \) for \( 1 \leq i \leq n \), \( a_{i,i-1} = a_{i,2n-i+2} = k \), for \( 2 \leq i \leq n \) (\( k \) is a positive constant), and 0 otherwise.
   (i). Compute the rank of \( A \).
   (ii). Find a basis for the row space of \( A \), the column space of \( A \) and the nullspace of \( A \).
   (iii). Does the system \( Ax = b \) have a solution if \( b \) is a vector all of whose entries are 1?
   If there is a solution, find all solutions. If there is no solution, find the least squares solution.
   (iv). Does the system \( A^T y = c \) have a solution, where \( c \) is a vector all of whose entries are 1? If there is a solution, find all solutions. If there is no solution, find the least squares solution.

6. Let \( A \) be a \( n \times n \) matrix. Prove that there exists a \( n \times n \) matrix \( B \) such that \( AB = 0 \) and \( \text{rank}(A) + \text{rank}(B) = n \).