**General Comprehensive Exam August 2012**
**Probability** (540)

Note: Closed book exam. Calculaters allowed.
Totally 6 indexed questions in 6 pages (you may use the back sides in case needed).
A table of probability distributions are given at the end.
Sufficient writing justifications are required for full credit.

Name_________________

1. (15 points) Assume the prevalence of HIV in a population is 2%, meaning that the chance that any randomly selected person has HIV is 0.02. A certain test for HIV has sensitivity 99.7%, meaning that if a person is HIV+, the probability that he/she will be tested as positive is 0.997. At the same time, this test for HIV also has specificity 98.5%, meaning that if a person is HIV−, the probability that he/she will be tested as negative is 0.985. Question: If someone tests positive, what is the chance the person really is HIV+? (Notice: If you didn't bring a calculator, it is ok to just give out the formula of the numbers without the final calculated result.)

2. Let $X$ have a $Uniform\left(-\frac{1}{2}, \frac{1}{2}\right)$ distribution and let $Z$ have a $Uniform(0, \frac{1}{2})$ distribution. $X$ and $Z$ are independent. Let $Y = X^2 + Z$.

(10 points) Calculate $Cov\,(X, Y)$.
(5 points) Are $X$ and $Y$ independent? Why or why not?

3. (20 points) Let $X_1, X_2, ..., X_n \sim Exponential\,(\beta)$. Show that $\sum_{i=1}^{n} X_i \sim Gamma\,(n, \beta)$. (Hint: You may directly use the information from the table of probability distributions given at the end.)

4. (20 points) Let random variables $X$ and $Y$ have joint pdf $f(x,y) = 1$, $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find the pdf of $Z = X + 2Y$.

5. Let $\sqrt{n}X_n \sim N(0, 1)$.

(10 points) Show that $X_n$ converges in probability to 0, as $n \to \infty$.
(10 points) Show that $X_n$ converges in distribution to a point mass at 0 (i.e., a distribution with probability 1at point 0), as $n \to \infty$.

6. (10 points) Prove Mill's Inequality: Let $Z \sim N(0,1)$. Then for any constant $t > 0$,

$$P\left(|Z| > t\right) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}.$$

Table of Distributions

| Distribution | PDF or probability function | mean | variance | MGF |
| --- | --- | --- | --- | --- |
| Point mass at $a$ | $I(x=a)$ | $a$ | $0$ | $e^{at}$ |
| Bernoulli$(p)$ | $p^x(1-p)^{1-x}$ | $p$ | $p(1-p)$ | $pe^t+(1-p)$ |
| Binomial$(n,p)$ | $\binom{n}{x}p^x(1-p)^{n-x}$ | $np$ | $np(1-p)$ | $(pe^t+(1-p))^n$ |
| Geometric$(p)$ | $p(1-p)^{x-1}I(x\geq 1)$ | $1/p$ | $\frac{1-p}{p^2}$ | $\frac{pe^t}{1-(1-p)e^t}\ \left(t<-\log(1-p)\right)$ |
| Poisson$(\lambda)$ | $\frac{\lambda^x e^{-\lambda}}{x!}$ | $\lambda$ | $\lambda$ | $e^{\lambda(e^t-1)}$ |
| Uniform$(a,b)$ | $I(a<x<b)/(b-a)$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{bt}-e^{at}}{(b-a)t}$ |
| Normal$(\mu,\sigma^2)$ | $\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ | $\mu$ | $\sigma^2$ | $\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$ |
| Exponential$(\beta)$ | $\frac{e^{-x/\beta}}{\beta}$ | $\beta$ | $\beta^2$ | $\frac{1}{1-\beta t}\ (t<1/\beta)$ |
| Gamma$(\alpha,\beta)$ | $\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$ | $\alpha\beta$ | $\alpha\beta^2$ | $\left(\frac{1}{1-\beta t}\right)^\alpha\ (t<1/\beta)$ |
| Beta$(\alpha,\beta)$ | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | $1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$ |
| $t_\nu$ | $\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\left(1+\frac{x^2}{\nu}\right)^{(\nu+1)/2}}$. | $0$ (if $\nu>1$) | $\frac{\nu}{\nu-2}$ (if $\nu>2$) | does not exist |
| $\chi^2_p$ | $\frac{1}{\Gamma(p/2)2^{p/2}}x^{(p/2)-1}e^{-x/2}$ | $p$ | $2p$ | $\left(\frac{1}{1-2t}\right)^{p/2}\ (t<1/2)$ |