WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540, August, 2013

Prove all results. You may quote standard results proved in classes. Each part counts for 10 points.

1. Consider a population of values $x_1, x_2, \ldots x_n$. A sample of size n is selected at random with replacement.

   (a) What is the probability x_1 appears in the sample?

   (b) Asymptotically, as $n \to \infty$, what proportion of the x_i are left out of the sample?

2. Suppose X has a standard Cauchy distribution: i.e., has pdf

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

   Find the distribution of $Y = 1/X$.

3. Recall that for two random variables, X and Z, the best predictor, in the mean square sense, of X given Z is $E(X|Z)$. Suppose $X, Y \stackrel{iid}{\sim} U(0,1)$. Find the best predictor, in the mean square sense, of X given $Z = XY$.

4. You have a coin which you are not sure is fair. Consider the following scheme: *Toss the coin twice. If both tosses give the same results, start again. The first time two successive tosses give different results (i.e., HT or TH), take the result of the second toss.*

   (a) Show that the scheme described above results in the outcome of a single toss of a fair coin.

   (b) What is the expected number of coin tosses you will have to make under this scheme?

   (c) Devise a similar scheme, based on 5 (rather than 2) coin tosses of a possibly biased coin, which can give a random number uniformly distributed on the integers 1 through 10.

5. Let $X_n,\ n = 1, 2, \ldots$ be a sequence of random variables.

   (a) Show that if there exist positive constants K and M such that $|X_n| \leq K,\ n = 1, 2, \ldots$ and $var(X_n) \geq M,\ n = 1, 2, \ldots$, then X_n cannot converge in probability to 0.

   (b) Is the result still true if either or both inequalities are omitted from the problem statement? Prove or give counterexamples.

6. Suppose $U_i \stackrel{iid}{\sim} U(0,1),\ i = 1, 2, \ldots$, and let $s \in (0,1]$ be a constant, and let $N = \min\{n : \sum_{i=1}^n U_i > s\}$. Show that $P(N > n) = s^n/n!$.