Graduate Comprehensive Examination

Department of Mathematical Sciences

MA540, Probability and Mathematical Statistics I

January 13, 2014

Answer ALL Questions in Two hours.
The questions are equally weighted.
You must score at least 70% to pass this test.

GOOD LUCK!

1. Consider the following two-player game: A coin with probability .7 of heads is to be tossed. This probability is known to both participants. Player 1 chooses the sequence HHH. Player 2 is to choose a different three outcome sequence. The coin is tossed repeatedly until one of the two sequences occurs, and the winner is the player who chose the winning sequence. What sequence should player 2 choose to maximize his/her probability of winning, and what is the probability of winning with that sequence?

2. Let $X \sim$ Binomial(n, p) and independently $Y \sim$ Binomial$\{m, \frac{p}{(1-p)\rho+p}\}, \rho > 0$. Find a form for the $Pr(a \leq X \leq b \mid X+Y=t)$, where a, b, t are all integers.

3. A value X is obtained as follows: An integer $Y \sim Poisson(\lambda)$ is selected. Then X is taken to be uniformly distributed on the interval $[Y, Y+1)$.

   (a) Is X continuous, discrete or mixed? Obtain its pdf, pmf or mixed distribution function as appropriate.

   (b) Find the mean and variance of X.

4. A random variable X having density function $f(x;\lambda) = 2\phi(x)\Phi(\lambda x), \quad -\infty < x < \infty$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the $N(0,1)$ pdf and cdf, respectively, is said to have skew normal distribution with parameter λ, denoted $X \sim SN(\lambda)$.

   (a) Show that $f(x;\lambda)$ is a pdf.

   (b) Show that $X \sim SN(0)$ implies $X \sim N(0,1)$.

5. Let $\{X_n\}$ be a sequence of random variables with the property that

$$|X_n| \leq Y \quad \text{for} \quad n \geq 1, \quad \text{where} \quad E(Y^2) < \infty.$$

Prove that

$$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0 \text{ for all } \epsilon > 0 \text{ implies } E\{(X_n - X)^2\} \to 0 \text{ as } n \to \infty.$$

6. Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$, a probability density function with mean, μ, and finite variance, σ^2. Let X_i denote the i^{th} order random variable. Prove that

$$\mu = \frac{1}{n}\sum_{i=1}^{n} \mu_{(i)}, \quad \mu_{(i)} = E(X_{(i)}), i = 1, \ldots, n.$$

Write down a similar formula for variances.