Graduate Comprehensive Examination

Department of Mathematical Sciences

MA540, Probability and Mathematical Statistics I

January 19, 2017

# 1. Let \( X, Y \sim \text{Gamma}(1, 1) \) and \( S = \frac{X}{p} - \frac{Y}{1-p} \), \( 0 < p < 1 \). Find the \( p \)th quantile of \( S \). What happens when \( p = 1/2 \)?

# 2. Suppose \( Z \sim \text{Beta}(\alpha, \beta + \gamma) \). Show that \( Z \) can be written as \( Z = XY \), where \( X \) and \( Y \) are independent with \( X \sim \text{Beta}(\alpha, \beta) \) and \( Y \sim \text{Beta}(\alpha + \beta, \gamma) \).

# 3. Let \( Z | X = x, \delta \sim \text{Normal}(\delta x, 1 - \delta^2) \), \( | \delta | < 1 \) and \( X \sim \text{Normal}(0, 1) \), \( x > 0 \) (half normal). Show that \( f(z) = 2\phi(x)\Phi(\lambda z) \), where \( \lambda = \delta / \sqrt{1 - \delta^2} \) and \( \phi(\cdot) \) and \( \Phi(\cdot) \) are respectively the pdf and cdf of the standard normal random variable. Find the moment generating function of \( Z \).

# 4. Suppose that \( X, Y | Z \sim \text{Bernoulli}(Z) \) and \( Z \sim \text{Beta}(\alpha, \beta) \).
   (a) Find the expectation of \( X \)
   (b) Find the Variance of \( X \).
   (c) Find covariance of \( X \) and \( Y \).
   (d) Show that \( X \) and \( Y \) are identically distributed.

# 5. Let \( X_n \sim \text{Poisson}(n\lambda) \) where the positive integer \( n \) is large and \( 0 < \lambda \).
   (a) Find the limiting distribution of \( \sqrt{n} \left( \frac{X_n}{n} - \lambda \right) \).
   (b) Find the limiting distribution of \( \sqrt{n} \left[ \frac{X_n}{n} - \sqrt{\lambda} \right] \).

# 6. Show that if \( U \) and \( V \) are independent uniform \((-1/2, 1/2)\) variables and \( U^2 + V^2 \leq 1/4 \), then \( U/V \) is a Cauchy variate.