Graduate Comprehensive Examination

Department of Mathematical Sciences

MA540, Probability and Mathematical Statistics I

May 19, 2008

TIME: Two Hours

Answer ANY SIX Questions in Section A and

ANY FOUR Questions in Section B.

Section A is worth 48% and Section B 52%.

You must score at least 70% to pass this test.

GOOD LUCK!

Section A: Answer ANY SIX Questions.

1. Consider the probability space $(\Omega, \mathcal{A}, P)$. Let $A_1, A_2, \ldots$ form a partition of Ω with $A_i \in \mathcal{A}$ and $P(A_i) > 0$. Show that $P(B) = \sum_{i=1}^{\infty} P(B \mid A_i)P(A_i)$.

2. An urn contains b blue balls and r red balls. A ball is taken at random, its color is noted, and two balls of the same color are returned to the urn. The process is repeated. Given that the second ball is red, what is the probability that the first ball is blue?

3. Each, of Tom and Mary, tosses a fair coin n times indpendently. Find the probability that the number of tails Tom gets is the same as the number of heads Mary gets. [Simplify your answer.]

4. Let $f(x) = e^{-x}/(1+e^{-x})^2, \ -\infty < x < \infty$. Find $E[\{f(X)\}^k]$ where k is a nonnegative integer. Show also that $E[\{f(X)\}^k] \leq 2^{-2k}$.

5. Suppose $f(x, y) = 3!, \ 0 \leq y \leq x(1-x), 0 < x < 1$, and $f(x, y) = 0$ otherwise. Find the conditional density functions of $x \mid y$ and $y \mid x$.

6. Construct two random variables X and Y which are uncorrelated and dependent.

7. Use the Central Limit Theorem to prove that

$$2 \lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = 1.$$

Section B: Answer ANY FOUR Questions.

\# 8. Let $X_1, X_2, X_3 \stackrel{iid}{\sim}$ Exponential(1), and let

$$Y_1 = \frac{X_1}{X_1 + X_2 + X_3} \text{ and } Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}.$$

Find the joint density of Y_1 and Y_2 and find their marginal densities.

\# 9. Let X_1, X_2, X_3, X_4 be continuous random variables with joint density $\pi(x_1, x_2, x_3, x_4)$; all marginal and conditional densities have ranges in $(-\infty, \infty)$. An approximation (A) to the joint density $\pi(x_1, x_2, x_3, x_4)$ is

$$\pi_A(x_1, x_2, x_3, x_4) = \pi_1(x_1, x_4)\pi_2(x_2 \mid x_1, x_4)\pi_3(x_3 \mid x_2, x_4),$$

where the $\pi_1(x_1, x_4)$, $\pi_2(x_2 \mid x_1, x_4)$ and $\pi_3(x_3 \mid x_2, x_4)$ are obtained under $\pi(x_1, x_2, x_3, x_4)$. Show that $\pi_A(x_2, x_3, x_4) = \pi(x_2, x_3, x_4)$, and $\pi_A(x_1) = \pi(x_1)$. What simple condition is needed to ensure that $\pi_A(x_1, x_2, x_3, x_4) = \pi(x_1, x_2, x_3, x_4)$?

\# 10. Find conditions on a, b and c such that $\int_{-\infty}^{\infty} x^c e^{ax}/(1+e^x)^b dx < \infty$.

\# 11. Let $X \sim$ Binomial(n, p), and let $B_n = (X - np)/\sqrt{np(1-p)}$. Use the moment generating function of B_n directly to show that B_n converges in distribution to a standard normal random variable.

\# 12. Let $X_1, \ldots, X_n$ be independent and identically-distributed Beta(2, 1) random variables and let $X_{(1)} = \min(X_1, \ldots, X_n)$. Letting $f_T(t) = 2te^{-t^2}$, $t \geq 0$ and $f_T(t) = 0$ otherwise, show that

$$X_{(1)} \xrightarrow{as} 0 \quad \text{and} \quad \sqrt{n}X_{(1)} \xrightarrow{d} T.$$