General Comprehensive Exam Spring 2012
Probability

Note: A table of distributions are given at the end.
Sufficient justifications are required for full credit.

Name_________________

1. (10 points) Let a random sample $X_1, ..., X_n$ follow a location-scale family distribution with probability density function (pdf) $\frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right)$ for some proper function $g(\cdot)$. μ is the location parameter, σ is the scale parameter. Another random sample $Z_1, ..., Z_n$ are i.i.d. with pdf $g(z)$, and the sampling distribution of sample mean $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ has pdf $f(z)$. What is the pdf for $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$?

2. (15 points) Two players, A and B, alternatively and independently flip a coin and the first player to obtain a head wins. Suppose that $P\,(\text{head}) = p$ and player A flips first. What is the probability that A wins?

3. Suppose the distribution of Y, conditional on $X = x$, is $N\left(x, x^2\right)$ and that the marginal distribution of X is uniform$(0, 1)$.

- (7 points) Calculate $Var\left(Y\right)$.
- (8 points) Calculate $Cov\left(X, Y\right)$.

4. (15 points) Let X have uniform distribution on $(-1, 2)$. Find the pdf of $Y = X^2$.

5. (15 points) Let X_1, ..., X_n be a random sample from population distribution $N(\theta, \tau^2)$. The sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that $\bar{X}_n$ converges in probability to θ.

6. (15 points) Suppose that X and Y have joint density function

$$f(x,y) = \begin{cases} x+y & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find $P(X < 1/4 | Y = 1/3)$.

7. (15 points) Let $X \sim Poisson\,(\lambda)$ and $Y \sim Poisson\,(\mu)$ and assume that X and Y are independent. What is the distribution of $X + Y$? Prove it.

Table of Distributions

Distribution	PDF or probability function	mean	variance	MGF
Point mass at a	$I(x=a)$	a	0	e^{at}
Bernoulli(p)	$p^x(1-p)^{1-x}$	p	$p(1-p)$	$pe^t+(1-p)$
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$(pe^t+(1-p))^n$
Geometric(p)	$p(1-p)^{x-1}I(x\geq 1)$	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$ $(t<-\log(1-p))$
Poisson(λ)	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform(a,b)	$I(a<x<b)/(b-a)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal(μ,σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2	$\exp\left\{\mu t+\frac{\sigma^2t^2}{2}\right\}$
Exponential(β)	$\frac{e^{-x/\beta}}{\beta}$	β	β^2	$\frac{1}{1-\beta t}$ $(t<1/\beta)$
Gamma(α,β)	$\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha$ $(t<1/\beta)$
Beta(α,β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
t_ν	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\left(1+\frac{x^2}{\nu}\right)^{(\nu+1)/2}}$.	0 (if $\nu>1$)	$\frac{\nu}{\nu-2}$ (if $\nu>2$)	does not exist
χ^2_p	$\frac{1}{\Gamma(p/2)2^{p/2}}x^{(p/2)-1}e^{-x/2}$	p	$2p$	$\left(\frac{1}{1-2t}\right)^{p/2}$ $(t<1/2)$