Graduate Comprehensive Examination

Department of Mathematical Sciences

MA541, Probability and Mathematical Statistics II

January 14, 2014

Answer ALL Questions in Two hours.
The questions are equally weighted.
You must score at least 70% to pass this test.

GOOD LUCK!

1. If $X_1, \ldots, X_n$ is a random sample from an exponential distribution with mean θ, obtain the maximum likelihood estimator of $\delta = E(X^{-1/2})$.

2. Let $X_1, \ldots, X_n \mid \theta \overset{iid}{\sim} \text{Uniform}(0, \theta)$, $\pi(\theta) = 1, \theta > 0$. Find the Bayes' estimator of θ under squared error loss. Is it unbiased? Is it asymptotically unbiased? Is it consistent? Explain.

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Normal}(\mu_1, \sigma^2)$ and independently $X_{n+1}, \ldots, X_{n+m} \overset{iid}{\sim} \text{Normal}(\mu_2, \sigma^2)$, where $\mu_1 \leq \mu_2$. Find the maximum likelihood estimators of μ_1, μ_2, σ^2. Write down a likelihood ratio test statistic for $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 < \mu_2$.

4. Let $X_{(k)}$ denote the k^{th} order statistic from a random sample of size n from a distribution with strictly increasing cdf F, and suppose $(X_{(i)}, X_{(i+j)})$, for some $1 \leq i < i + j \leq n$, is a prediction interval for a new observation. Find the confidence level.

5. Let X be a single observation from

$$f(x \mid \theta) = \theta(\theta + 1)(1 - x)x^{\theta - 1}, \; 0 < x < 1, \; \theta > 0.$$

Explain why there is a uniformly most powerful test of $H_0 : \theta \leq 1$ versus $H_a : \theta > 1$, and determine the test completely if its size is α. Deduce the power function of this test.

6. Let T_n be the largest order statistic of a random sample of size n from a $U(0, \theta)$ distribution.

   (a) Show that $V_n = n(\theta - T_n)/\theta$ converges in distribution to an exponential distribution with mean 1.

   (b) Use the result in (a) to give a formula for an approximate level $1 - \alpha$ equal-tailed (in probability) confidence interval for θ.