Graduate Comprehensive Examination

Department of Mathematical Sciences

MA541, Probability and Mathematical Statistics II

January 20, 2017

1. A random sample of n adults is taken from a large population to estimate the proportion, π, of adults responding 'yes' to item, A, and each adult is asked to respond truthfully. Each respondent is asked to toss a biased coin (probability of heads is p, known). If the coin comes up heads, the respondent is asked to answer A. If the coin comes up tails, each respondent is asked to toss a second biased coin (probability of heads is q, known). If the second coin comes up heads, answer A and if it comes up tails, answer the opposite of A. Suppose there are y 'yeses' among the n adults. Find the maximum likelihood estimator of π and find its standard error.

2. Let $X_1, \ldots, X_n \mid \theta \overset{ind}{\sim} \text{Uniform}(0, \theta), \theta > 0$.
(a) Find the $100(1-\alpha)\%$ shortest confidence interval of θ.
(b) Find an unbiased estimator of θ based on $R = X_{(n)} - X_{(1)}$, where $X_{(1)}$ and $X_{(n)}$ are respectively the smallest and largest order statistics. Find $\text{Var}(R)$.

3. Let $A = E_X[F(a + bX)]$, where a and $b \neq 0$ are contants and $F(\cdot)$ is the cdf of X. Show that $A = P(Z - bX \leq a)$, where Z has cdf, $F(\cdot)$, and is independent of X. Deduce that $\Phi\{\sqrt{n/(n-1)}(\bar{X} - a)\}, n > 1$, is the minimum variance unbiased estimator of $\Phi(\mu - a)$, where $X_1, \ldots, X_n \mid \mu \overset{ind}{\sim} \text{Normal}(\mu, 1)$.

4. Let X_k's be independent random variables with X_k distributed as Uniform on the interval $(-k\theta, k\theta + 2k)$, $\theta > 0$ with $k = 1, \ldots, n$. One is asked to work with all the observations $X_1, \ldots, X_n$.

(a) Obtain the complete and sufficient statistic for θ. What is the MLE of θ? What is the MLE for θ^2?

(b) Derive the UMVUE of θ. Derive the UMVUE of θ^2.

(c) Suppose that θ has the prior distribution given by

$$\pi(\theta) = \theta^{-2} I(1 < \theta < \infty).$$

Assuming squared error loss, obtain the Bayes estimate of θ.

5. Let $X_1, \ldots, X_n$ be a random sample from the distribution with mass function,

$$f(x) = \begin{cases} \frac{3(1-\theta)}{3-\theta}, & x = 1 \\ \\ \frac{2\theta}{3-\theta}, & x = 2 \end{cases}$$

where $0 < \theta < 1$.

(a) Find the method of moments estimator of θ.

(b) Find the MLE of θ. How does it compare to the MM estimator from part (a)?

6. Suppose we want to test $H_0 : \theta = 0$ versus $H_1 : \theta = 1$ using a single observation X from a $N(\theta, 1)$ distribution.

(a) Derive a formula for a most powerful level α test.

(b) Now derive the likelihood ratio test of the same hypotheses. Is it the same as the test in part (a)?