Graduate Comprehensive Examination

Department of Mathematical Sciences

MA541, Probability and Mathematical Statistics II

Answer ALL questions in THREE hours.

May 9, 2017

1. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Consider the class of estimators,

$$T_\omega = \omega \bar{X} + (1 - \omega)S^2, 0 \leq \omega \leq 1,$$

where $\bar{X}$ is the sample mean and $S^2$ the sample variance.

   a. What is the size of this class of estimators and what key property does it have?

   b. Find the minimum variance of $T_\omega$ as a function of $\omega$.

   c. Show that $E(S^4) > \lambda/n + \lambda^2$.

2. Let $X_1, \ldots, X_n \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)$. Consider the following interval estimator $(\bar{X} - kS, \bar{X} + kS)$ of $\mu$, where $k$ is a positive constant, $\bar{X}$ is the sample mean and $S^2$ is the sample variance.

   a. Find the distribution of $S$.

   b. Find the probability content of the interval conditional on $S = s$.

   c. Find the probability content of this interval, without conditioning on $S$, in its simplest form.

3. Let $X_1, \ldots, X_{2n} \mid \theta \sim \text{Uniform}(0, \theta), \theta > 0$. Suppose the first $n$ values are observed and the next $n$ values are missing. Let $T$ denote the largest observation among the missing values.

   a. Find a pivotal quantity that is an ancillary statistic.

   b. Find the distribution of the pivotal quantity.

   c. Find the $100(1 - \alpha)\%$ shortest prediction interval for $T$. 


4. An item on a questionnaire asks $n$ respondents to report the value of one of two positive random variables, $X$ or $SX$. Each respondent actually reports $Z = SYX$, where $Y \sim \text{Bernoulli}(p)$, $p$ known. Here, $X$ and $S$ are independent and the distribution of $S$ is completely known, $E(S) = 1$ and $\text{Var}(S) = \mu$. Inference is required about the mean, $\mu$, of $X$ with $\text{Var}(X) = \sigma^2$ known. Let $\hat{\mu}$ denote an unbiased estimator of $\mu$.

a. Obtain a form for $\hat{\mu}$.

b. Find $\text{Var}(\hat{\mu})$.

5. Suppose $Y$ is a random variable from a Weibull distribution with shape parameters $\lambda$ and scale parameter $\theta$, i.e., $Y \sim \text{WB}(\theta, \lambda)$, with its pdf as

$$f(y \mid \theta, \lambda) = \frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp\left(-\left(\frac{y}{\theta}\right)^\lambda\right), \quad y > 0,$$

where $\theta > 0$ and $\lambda > 0$.

a. Show that if $\lambda = \lambda_0 > 0$ is known, then $\text{WB}(\theta, \lambda_0)$, $\theta > 0$ is a member of the exponential family. What if $\lambda$ is unknown? Justify.

b. Derive the MLE of the parameter $\theta$ when $\lambda = \lambda_0$ is known. For this case, obtain the asymptotic distribution of the MLE of $\theta$.

c. Derive the MLE of the parameters $\theta$ and $\lambda$ when both are unknown.

6. Consider the following pdf

$$f(x, y \mid \sigma) = \frac{\sigma^2}{\pi \sqrt{3}} \exp\left\{-\frac{2\sigma^2}{3} \left[ \frac{(x-1)^2 + (y-2)^2 - (x-1)(y-2)}{3} \right] \right\} \quad \text{for } (x, y) \in \mathbb{R}^2,$$

where $\sigma$ is the unknown parameter. Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. with common pdf $f(x, y \mid \sigma)$.

a. Find the form of the UMP level $\alpha$ test for

$$H_0 : \sigma \leq \sigma_0 \text{ versus } H_1 : \sigma > \sigma_0.$$ 

b. Show that there is no UMP level $\alpha$ test for

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0$$

with $0 < \alpha < 1$ fixed.

c. Derive the level $\alpha$ likelihood ratio test for

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0$$

in its simplest implementable form, with $0 < \alpha < 1$ fixed.