Graduate Comprehensive Examination Spring 2012

Department of Mathematical Sciences

April, 2012

MA541, Probability and Mathematical Statistics II

You are required to answer ANY SIX questions in TWO hours.

GOOD LUCK!

1. Let Z_1 and Z_2 be independent standard normal random variables. By using the result on the sample mean and the sample variance, give an argument to show that $(Z_1 + Z_2)/\sqrt{2}$ and $(Z_1 - Z_2)/\sqrt{2}$ are independent. [3 points]. If in addition $Z_1 > Z_2$, find the joint density of $(Z_1,\ Z_2)$. [2 points]

2. Let $X_1, \ldots, X_n$ be a random sample from the probability density function

$$f_{X_i}(x \mid \theta) = \begin{cases} e^{i\theta - x}, & x \geq i\theta \\ 0, & x < i\theta. \end{cases}$$

Prove that $T = \min\{X_1, X_2/2, \ldots, X_n/n\}$ is a sufficient statistic. [5 points]

3. Let $X_1, \ldots, X_n$ be a random sample from the uniform probability density function on $(0,\ \theta)$. Let $R = \frac{X_{(1)}}{X_{(n)}}$ where $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Show that $E(R) = E(X_{(1)})/E(X_{(n)}) = n^{-1}$. [5 points]

4. Suppose that

$$(Y_1, Y_2, Y_3) \mid \theta \sim \text{multinomial}\{n,\ \ (p_1, p_2, p_3)\},\ \ Y_1 + Y_2 + Y_3 = n,\ \ Y_1,\ Y_2,\ Y_3 \geq 0,$$

where $p_1 = \theta$, $p_2 = 1/2$ and $p_3 = 1/2 - \theta$ with $0 < \theta < 1/2$, and suppose that a scientist can only observe Y_1 and $Y_2 + Y_3$. Use the expectation-maximization algorithm to find the MLE of θ. [5 points]

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the pdf

$$f(x \mid \theta) = e^{-(x-\theta)},\ \theta < x < \infty,\ \ -\infty < \theta < \infty.$$

Find the maximum likelihood estimator (MLE) for θ, and write down the MLE for $e^{-2\theta}$. [5 points]

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(0, \theta)$, and suppose that $p(\theta) = 1,\ \theta > 0$. Find the Bayes estimator for θ under squared error loss. [5 points]

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x \mid \gamma) = \gamma^{-1} x^{\gamma^{-1}-1},\ 0 < x < 1$. Use the Cramer-Rao bound to find the best unbiased estimator, $\hat{\gamma}$, of γ. [3 points] Describe the asymptotic distribution of $\hat{\gamma}$ [2 points]

8. Suppose X has the probability density function $f(x - \theta)$. Prove and discuss an important property of the power function for testing $H_0:\ \theta \leq 0$ versus $H_1:\ \theta > 0$ with rejection region $R = \{x : x > c\}$? [5 points]

9. Let X be a single observation from

$$f(x \mid \theta) = \theta(\theta+1)(1-x)x^{\theta-1},\ 0 < x < 1,\ \theta > 0.$$

Explain why there is a uniformly most powerful test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, and determine the test completely if its size is α. [5 points]

#10. The sampling distribution of a statistic used to estimate a parameter θ is asymmetric and unimodal. Discuss why an equal-ordinate 95% confidence interval for θ is shorter than an equal-tail 95% confidence interval. [2 points] Suppose that $T - \theta$, has a sampling distribution which is the unit exponential. Find the shortest $100(1-\alpha)$ confidence interval for θ. [3 points]