

WORCESTER POLYTECHNIC INSTITUTE

FOURTEENTH ANNUAL INVITATIONAL MATH MEET

OCTOBER 25, 2001

TEAM EXAM QUESTION SHEET

DIRECTIONS: Please write your answers on the Team Answer Sheet provided. This part of the contest is 45 minutes. Each correct answer to questions 1-14 is worth 3 points. Calculators **MAY NOT** be used.

- 1 Consider an equilateral triangle whose side is a units long and has a circle inscribed in it. What is the area of the circle?

$$\pi a^2/12$$

- 2 A triangle in three dimensions has its vertices at $(7, 3, 13)$, $(12, 3, 13)$, and $(c+7, 15, 13)$. Find a value of c so that this a right triangle.

$$c = 5 \text{ or } c = 0$$

- 3 What is the 19th term in the following sequence:

$$\frac{1}{362}$$

$$\{1/2, 9/5, 1/10, 33/17, 1/26, 73/37, \dots\} ?$$

- 4 Fermat's Little Theorem states that if p does not divide a then $a^{p-1} \equiv 1 \pmod{p}$.

Use it to simplify $12^{3382} \pmod{261}$

("mod p " means that the numbers differ by any multiple of p ; so $33 \equiv 5 \pmod{7}$ because $33 - 5 = 4 * 7$. Similarly $40, 47, \text{ and } 54 \equiv 5 \pmod{7}$).

$$144$$

- 5 A parabola has the equation $y = x^2/8$. A ray comes down parallel to the y axis, strikes the parabola and reflects off of it. Where does the reflected ray first cross the y axis?

$$(0, 2) \text{ or } y = 2$$

- 6 Find the sum

$$14496$$

$$195 + 196 + 197 + 198 + \dots + 257 + 258$$

- 7 If one expands $(a + b)^{19}$ and then picks one of the coefficients at random, what is the probability that coefficient will be a prime number? Assume 1 is not a prime number for sake of discussion.

$$2/20 \text{ or } 1/10$$

- 8 The following number is in base **seven**. Convert it to decimal (base 10)

$$2075\frac{16}{49}$$

6023.22

- 9 Two pulleys are connected by a belt. The radii of the pulleys are 3 cm and 15 cm, and the distance between their centers is 24 cm. Find the total length of the belt needed to connect the pulleys. Assume there is no slack in it.

$$24\sqrt{3} + 22\pi cm$$

- 10 Simplify the sum (your answer should be in terms of n)

$$3^n$$

$$\sum_{k=0}^n 2^k \binom{n}{k}$$

- 11 A is an $n \times n$ matrix. Each of the rows in A contains the integers 1 through n in a random order. B is the $n \times 1$ column matrix whose entries are all 1. What is the matrix product AB ?

$$\text{an } n \times 1 \text{ column where all entries are } n(n+1)/2$$

- 12 f is a cubic polynomial function. It is known that the graph of f passes through the points $(0, 1)$, $(1, 10)$, $(3, 844)$ and $(-1, -8)$. What value is $f(2)$?

$$223 \quad (f(x) = 34x^3 - 25x + 1)$$

- 13 Factor the following

$$(x - 1)(x - 3)^2(x + 4)$$

$$x^4 - 3x^3 - 13x^2 + 51x - 36$$

- 14 If n is a positive integer, what is $2^n - 1$ in binary?

$$n \text{ 1's or } \underbrace{1111 \dots 1}_2$$

$n \text{ times}$