WORCESTER POLYTECHNIC INSTITUTE
TWENTY-FOURTH ANNUAL INVITATIONAL MATH MEET
OCTOBER 19, 2011
TEAM EXAM QUESTION SHEET WITH ANSWERS

DIRECTIONS: Please write your answers on the **Team Answer Sheet** provided. This part of the contest is 45 minutes. All 14 problems are counted equally. Calculators **MAY NOT** be used.

1. A circle is inscribed in a square of side m, then a square inside that circle, then a circle inside the latter square and so on. If S_n is the sum of the areas of the first n such circles so inscribed, then as n grows without bound, what does S_n approach?

   Ans: $\frac{\pi m^2}{2}$

2. If $S = 1! + 2! \cdots + 99!$ then what is the units value in S ?

   Ans: 3

3. The number $2^{48} - 1$ is exactly divisible by two numbers between 60 and 70. What are they?

   Ans: 63, 65

4. If $z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ in Cartesian form, find z^{15}.

   Ans: -1

5. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ then what is $f\left(\frac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$?

   Ans: $3f(x)$

6. The sum of all but one interior angles of a convex polygon is $2570°$**.** The remaining angle must be what?

   Ans: $130°$

7. The number of real solutions to

   $$\frac{X}{100} = \sin(X)$$

   is what?

   Ans: 63

8. In what follows, all matrices are $n \times n$. Given $A = QRS$ where $Q = S^{-1}$ and the entries of R satisfy $r_{ij} = \begin{cases} 0, & i \neq j \\ (-1)^{3i} & i = j \end{cases}$. What is A^{2p+1} where p is any prime number?

Ans: A

9. The greatest integer that will divide the three integers **13, 511**, **13,903** and **14, 589** and leave the same remainder is________.

Ans: 98

10. If the graph of $x^2 + y^2 = m$, where m > 0 , is tangent to that of $x + y = k$ what must k be ?

Ans: $\pm\sqrt{2m}$

11. A number n has 3 digits when expressed in base 7. When n is expressed in base 9 the digits are reversed. What is the middle digit?

Ans: 0

12. Let $S = 2 + 4 + 6 + \cdots + 2N = 2\,(1 + \cdots + N)$ where N is the smallest positive integer such that S > 1 million. Then the sum of the digits of N is what?

Ans: 1 (since $N = 1000$)

13. Let n be the number of ways that $10 can be changed into dimes and quarters, with at least one of each being used. Then what is n ?

Ans: 19

14. Find the 6^{th} root of -729 which lies in the third quadrant of the complex plane.

Ans: $\frac{-3\sqrt{3}}{2} - \frac{3}{2}i$