WORCESTER POLYTECHNIC INSTITUTE
31st INVITATIONAL MATH MEET
October 16, 2018
INDIVIDUAL EXAM QUESTION SHEET WITH ANSWERS

DIRECTIONS: Please write your answers on the Individual Answer Sheet provided. This part of the contest is 45 minutes long. Questions 1-4 are each worth 1 point. Questions 5-8 are each worth 2 points. Questions 9-11 are each worth 3 points. Calculators and other electronics MAY NOT be used.

Problem 1. A 2-dimension region described in polar coordinates is bound by:
   a) $\theta = \frac{\pi}{3}$  
   b) $\theta = \frac{3\pi}{4}$  
   c) $r \sin(\theta) = 4$  
   What is it’s area?

   Ans: $8 + \frac{8}{\sqrt{3}}$

Problem 2. Find $x$ that satisfies $x^2 + y\sqrt{xy} = 10$ and $y^2 + x\sqrt{xy} = 20$.

   Ans: $x = \frac{\sqrt{40}}{3}$

Problem 3. If $x^{11} + a^{11}$ is divided by $x + a$ what is the remainder?

   Ans: 0

Problem 4. Factor $1 + x^6$ as completely as possible over the Reals.

   Ans: $(1 + x^2)(x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1)$

Problem 5. Simplify as much as possible:

   $\sin^8(x) + 6\sin^4(x)\cos^4(x) + 4\cos^2(x)\sin^6(x) + 2 + \cos^8(x) + 4\sin^2(x)\cos^6(x)$

   Ans: 3

Problem 6. If $\sqrt{4 + x} + \sqrt{10 - x} = 6$. What is the value of $\sqrt{(4 + x)(10 - x)}$?

   Ans: 11

Problem 7. A cube is illustrated in the following diagram.

How many degrees measures the angle $< ABC$?
Problem 8. An isosceles right triangle is removed from each corner of a square piece of paper so that a rectangle remains. If the total area of the cut-off pieces is 200, what is the length of a diagonal of the rectangle?

Ans: 20

Problem 9. One circle has radius of 5 and center at (0, 5). A second circle has radius of 12 and center at (12, 0). What is the length of the radius of a third circle that passes through the center of the second circle and both points of intersection of the first two circles?

Ans: $\frac{13}{2}$

Problem 10. If $a$, $b$ and $c$ are all positive integer powers of 2 and it is given that $a^3 + b^4 = c^5$, what is the least possible value of $a + b + c$?

Ans: 352

Problem 11. Find all positive $x$ which satisfy

\[
\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x.
\]

Ans: $x = 1, x = 48