

WORCESTER POLYTECHNIC INSTITUTE

31st INVITATIONAL MATH MEET

October 16, 2018

INDIVIDUAL EXAM QUESTION SHEET WITH ANSWERS

DIRECTIONS: Please write your answers on the Individual Answer Sheet provided. This part of the contest is 45 minutes long. Questions 1-4 are each worth 1 point. Questions 5-8 are each worth 2 points. Questions 9-11 are each worth 3 points. Calculators and other electronics **MAY NOT** be used.

Problem 1. A 2-dimension region described in polar coordinates is bound by:
a) $\theta = \frac{\pi}{3}$ b) $\theta = \frac{3\pi}{4}$ c) $r \sin(\theta) = 4$ What is it's area?

Ans: $8 + \frac{8}{\sqrt{3}}$

Problem 2. Find x that satisfies $x^2 + y\sqrt{xy} = 10$ and $y^2 + x\sqrt{xy} = 20$.

Ans: $x = \frac{\sqrt{10}}{3}$

Problem 3. If $x^{11} + a^{11}$ is divided by $x + a$ what is the remainder?

Ans: 0

Problem 4. Factor $1 + x^6$ as completely as possible over the Reals.

Ans: $(1 + x^2)(x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1)$

Problem 5. Simplify as much as possible:

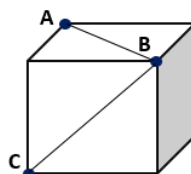
$$\sin^8(x) + 6 \sin^4(x) \cos^4(x) + 4 \cos^2(x) \sin^6(x) + 2 + \cos^8(x) + 4 \sin^2(x) \cos^6(x)$$

Ans: 3

Problem 6. If $\sqrt{4+x} + \sqrt{10-x} = 6$. What is the value of $\sqrt{(4+x)(10-x)}$?

Ans: 11

Problem 7. A cube is illustrated in the following diagram.



How many degrees measures the angle $\angle ABC$?

Ans: 60°

Problem 8. An isosceles right triangle is removed from each corner of a square piece of paper so that a rectangle remains. If the total area of the cut-off pieces is 200, what is the length of a diagonal of the rectangle?

Ans: 20

Problem 9. One circle has radius of 5 and center at $(0, 5)$. A second circle has radius of 12 and center at $(12, 0)$. What is the length of the radius of a third circle that passes through the center of the second circle and both points of intersection of the first two circles?

Ans: $\frac{13}{2}$

Problem 10. If a , b and c are all positive integer powers of 2 and it is given that $a^3 + b^4 = c^5$, what is the least possible value of $a + b + c$?

Ans: 352

Problem 11. Find all positive x which satisfy

$$\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x.$$

Ans: $x = 1, x = 48$