Real Analysis GCE May, 2019 Name:__________________________

Exercise 1:
Let V be a normed vector space and S a subset of V. Let S^c be the complement of S. Let x be in S and y be in S^c. The line segment $[x, y]$ is by definition the set

$$\{(1-t)x + ty : t \in [0,1]\}.$$

Show that the intersection of $[x, y]$ and ∂S is non empty, where ∂S is the boundary of S (by definition the boundary of S is the set of points that are in the closure of S and that are not in the interior of S).

Exercise 2:
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be a measurable function defined on X. Set

$$p_g(t) = \mu(\{x \in X : |g(x)| > t\}).$$

(i). If f is in $L^1(X)$ show that there is a constant $C > 0$ such that $p_f(t) \leq \dfrac{C}{t}$.
(ii). Find a measurable function h defined almost everywhere on $\mathbb{R}$ such that $\exists C > 0$, $p_h(t) \leq \dfrac{C}{t}$ and h is not in $L^1(\mathbb{R})$.

Exercise 3:
Let $\{f_n\} : [0,1] \to [0,\infty)$ be a sequence of functions, each of which is non-decreasing on the interval $[0,1]$. Suppose the sequence is uniformly bounded in $L^2([0,1])$. Show that there exists a subsequence that converges in $L^1([0,1])$.

Exercise 4:
Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ where $f_1(x) = \sqrt{x}$, $f_2(x) = \sqrt{x + \sqrt{x}}$, $f_3(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$, and in general $f_n(x) = \sqrt{x + \sqrt{x + \sqrt{\ldots + \sqrt{x}}}}$ with n roots.

1. Show that this sequence convereges pointwise on $[0,1]$ and find the limit function f such that $f_n \to f$.

2. Does this sequence converge *uniformly* on $[0,1]$? Prove or disprove uniform convergence.