GCE August 2020 – 502, Linear Algebra
No documents, no calculators allowed.
Attempt all questions.

1. Let V be an n-dimensional vector space over $\mathbb{C}$, and let $T : V \to V$ be a linear operator.
   (a) Define $\ker(T)$ and $\mathrm{Im}(T)$. State the Rank-Nullity Theorem.
   (b) Give an example of an operator T such that V is not the direct sum of the subspaces $\ker(T)$ and $\mathrm{Im}(T)$. (Hint: Consider the space of polynomials of degree $\leq n-1$, and let T be the differentiation operator.)
   (c) Prove that V **is** the direct sum of $\ker(T^n)$ and $\mathrm{Im}(T^n)$.
2. Prove that $A \in \mathrm{GL}_n(\mathbb{C})$ is invertible if and only if the minimum polynomial of A has a non-zero constant term. Under this condition, express A^{-1} as a polynomial function of A.
3. Let $f : M \mapsto \frac{1}{2}\left(M + M^\top\right)$ be an operator on $n \times n$ matrices.
   (a) Prove that f is linear, and that $f^2 = f$.
   (b) Show that all eigenvalues of f belong to $\{0, 1\}$.
   (c) Describe the eigenspaces of f.
4. Let V be $\mathbb{R}^n$ equipped with the standard inner product. For an arbitrary subspace U of V, let $U^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U\}$.
   (a) Show that $U \cap U^\perp = \{0\}$.
   (b) Show that $U \oplus U^\perp = V$.
   (c) Show $(U^\perp)^\perp = U$.
   (d) Which of these statements remain true over a field of positive characteristic?
5. Let V be the vector space of continuous, integrable functions $f : [-1, 1] \to \mathbb{R}$ equipped with inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{dx}$.
   (a) Prove that the only function satisfying $\langle f, f\rangle = 0$ is the zero function.
   (b) Find the projection of $f(x) = x^2 + 1$ onto the subspace $\langle 1, x\rangle$.
   (c) Compute the cosine of the angle between the functions $x^2 + 1$ and x with respect to the given inner product.
6. Let $A \in \mathbb{R}^{m\times n}$ and $B = \mathbb{R}^{n\times m}$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$