WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics - I
August, 2020

Note: The problems are arbitrarily ordered, not necessarily according to difficulty. Please show a clear logic of your solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Let $X_1, X_2 \ldots, X_n$ be *iid* from a normal distribution with mean μ and known variance σ^2. Let $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.
   1. Show that $\bar{X}$ and S^2 are independent.
   2. Find the limiting distribution of $\sqrt{n}(\bar{X}^3 - c)$ for an appropriate constant c.

2. (20 points) A student's Amazon Music playlist contains 100 songs, of which 10 are by the Beatles. Suppose all songs are randomly shuffled. Find the following probabilities. [Please give out proper rationals and formulas. You don't have to get the final numeric values.]
   (a) The probability that the first Beatles song heard is the fifth song played.
   (b) The probability that at least one of the first five songs played is a Beatles song.

3. (20 points) Let $X \sim Poisson(\lambda)$. Recall that the moment generating function of X is $\phi(s) = e^{\lambda(e^s-1)}$.
   1. Use $\phi(s)$ to find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.
   2. Show that $\mathrm{P}(X - \lambda \geq r) \leq \exp\left\{r - (\lambda + r)\log\left(\frac{\lambda+r}{\lambda}\right)\right\}$.
   (Hint: consider Markov's inequality: if Y is a *nonnegative* random variable and $a > 0$, then $\mathrm{P}(Y \geq a) \leq \frac{\mathrm{E}(Y)}{a}$.)

4. (20 points) Infectious diseases are sometimes modeled with a so called **SIR model** (the letters stand for Susceptible, Infected, and Recovered). People begin in class S, then possibly migrate to class I (i.e., become infected), and then to class R (i.e., recover); no other transitions are possible. In a simple version of the model, the ith individual begins in class S, waits a random amount of time $T_i \sim \exp(1/\lambda)$ before migrating to class I, then waits another random amount of time $U_i \sim \exp(1/\mu)$ before migrating to class R, with all the exponentially-distributed random variables T_i and U_i independent. Here a random variable $X \sim \exp(1/\lambda)$ if X has the pdf $f(x|\lambda) = \lambda\exp(-\lambda x)$ for $x > 0$.
   (a) Let N denote the number of Susceptibles at time 0 and let X_t be the number of these who become infected by time t. Find the probability distribution of X_t.
   (b) Let W_N be the length of time until the **last** of the N Susceptibles becomes infected. Find the probability density function for W_N.

(c) Let $Y_i = T_i + U_i$ be the total amount of time the ith Susceptible waits before joining class R. Find the probability distribution of Y_i under the (simplifying) assumption $\lambda = \mu$, explaining your reasoning.

5. (20 points) What nonzero distinct values of a, b, c will turn

$$f(x) = \exp(-ax^2 + 2bx - c), -\infty < x < \infty$$

into a normal probability density function?

6. (20 points) Let $X, Y, Z \overset{ind}{\sim} \text{Gamma}(1, \theta)$. Also, let

$$R = \frac{X}{X + Y + Z} \quad \text{and} \quad S = \frac{X + Y}{X + Y + Z}.$$

Find the joint probability density function of (R, S). Deduce that $S - R$ and R are identically distributed.