WPI Department of Mathematical Sciences 503 GCE

Name:______________________________

Exercise 1:
Let $E := [0,1] - S_{\mathbb{Q}} = [0,1] \cap (S_{\mathbb{Q}})^c$ where $S_{\mathbb{Q}} := \{x \in [0,1] \,|\, x = \sqrt{p}/q \text{ for some } p, q \in \mathbb{Z}^+\}$.
Prove or disprove: There exists a closed, uncountable subset $F \subset E$.

Exercise 2:
For x in $[-1,1]$ set $P_n(x) = c_n(1-x^2)^n$ where c_n is such that $\int_{-1}^{1} P_n = 1$.
(i). Show that there is a positive constant C such that $c_n \leq C\sqrt{n}$.
(ii). Let f be a real valued continuous function on $[0,1]$ such that $f(0) = f(1) = 0$. Set for x in $[0,1]$

$$f_n(x) = \int_0^1 P_n(x-t)f(t)dt.$$

Show that f_n is uniformly convergent to f.
Hint: Extend f to a function from $\mathbb{R}$ to $\mathbb{R}$ by zero.
(iii). Let g be in $L^1((0,1))$. Defining $g_n(x) = \int_0^1 P_n(x-t)g(t)dt$, is g_n uniformly convergent to g in $(0,1)$? Does g_n converge to g in $L^1((0,1))$?

Exercise 3:
Give an example of $f_n, f : \mathbb{R} \to [0,\infty)$ such that $f_n \in L^1(\mathbb{R})$ for every $n \in \mathbb{N}$, $f \in L^2(\mathbb{R})$, $f_n \leq f$ for every $n \in \mathbb{N}$, $f_n \to 0$ a.e., and $\int f_n \not\to 0$.