GCE: 503, Analysis and measure theory

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Exercise 1:

Let (X, ρ) be a metric space and K_n a sequence of compact subsets of X such that $K_{n+1} \subset K_n$. Set

$$d_n = \sup\{\rho(x,y) : x \in K_n, y \in K_n\}.$$

Assuming that d_n converges to zero show that $\bigcap_{n=1}^{\infty} K_n$ is a singleton.

Exercise 2:

(ii). Let $[a,b]$ be an interval in $\mathbb{R}$. If $\tilde{f}$ is continuous on $[a,b]$, g is differentiable on $[a,b]$ and monotonic, and g' is continuous on $[a,b]$, show that there is a c in $[a,b]$ such that

$$\int_a^b \tilde{f} g = g(a) \int_a^c \tilde{f} + g(b) \int_c^b \tilde{f}$$

Hint: Introduce $F(x) = \int_a^x \tilde{f}$ and integrate by parts.

(ii). Show that if g is as specified above and f is in $L^1([a,b])$, there is a c in $[a,b]$ such that

$$\int_a^b fg = g(a) \int_a^c f + g(b) \int_c^b f$$

Exercise 3:

Let $\{f_n\}$ be a sequence of functions $f_n : [0,1] \to \mathbb{R}$.

(i) Define equicontinuity for this sequence.

(ii) Show that if each f_n is differentiable on $[0,1]$ and $|f_n'(x)| \leq 1$ for all x in $[0,1]$ and $n \in \mathbb{N}$, the sequence is equicontinuous.

(iii) Suppose the sequence is uniformly bounded and that (ii) holds. Show that f_n has a subsequence which converges uniformly to a continuous function .

(iv) Show through an example that the limit may not be differentiable.

Exercise 4:

Let f be a lebesgue measurable function such that

$$\int_0^1 f(x) e^{Kx} dx = 0$$

for all $K = 1, 2, 3, ...$

Show that necessarily $f(x) = 0$ for almost every $0 \leq x \leq 1$.