GCE: 502, Linear Algebra

January 2020

No documents, no calculators allowed

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Exercise 1:
Prove that an m by n matrix A has rank at most r if and only if A can be expressed as a sum of r rank one matrices.

Exercise 2:
Let A be a $n \times n$ matrix. Prove that there exists a $n \times n$ matrix B such that $AB = 0$ and $rank(A) + rank(B) = n$.

Exercise 3:
Show that for any $n \times n$ real matrix A, $\sin A$ and $\cos A$ are well defined through their power series expansion, and prove that $(\cos A)^2 + (\sin A)^2 = I$, where I is the n by n identity matrix.

Exercise 4:
(i). Compute $\exp(tA)$ if $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and t is in $\mathbb{R}$.
(ii). Prove that, if $AB = BA$, then $\exp(A)\exp(B) = \exp(A + B)$.
(iii). Prove that, if A is *skew-symmetric* (i.e., $A^\top = -A$) then $\exp(A)$ is an orthogonal matrix.

Exercise 5:
Let A be an invertible n by n matrix. Show that there is a polynomial P with degree less or equal than $n - 1$ such that $A^{-1} = P(A)$.