GCE: 503, Analysis and measure theory

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Exercise 1:

Let E be a measurable subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ a measurable function. For a in $\mathbb{R}$, set $\omega_f(a) = |\{x \in E : f(x) > a\}|$ where $|\cdot|$ denotes the Lebesgue measure.

(i). If $f_k : E \to \mathbb{R}$ is a sequence of Lebesgue measurable, real-valued functions, such that $f_k \leq f_{k+1}$ and $f_k \to f$ almost everywhere, show that $\omega_{f_k} \leq \omega_{f_{k+1}}$ and $\omega_{f_k} \to \omega_f$.

(ii). Recall that f_k converges in measure to f if for all positive ϵ, $|\{x \in E : |f_k(x) - f(x)| > \epsilon\}|$ tends to zero as k tends to infinity.
If f_k converges in measure to f then show that $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a-\epsilon)$, and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\epsilon)$, for every $\epsilon > 0$.

(iii). If f_k converges in measure to f, show that $\omega_{f_k}(a) \to \omega_f(a)$ if ω_f is continuous at a.

Exercise 2:

(i). Define the sequence of functions $g_n : [0,1] \to \mathbb{R}$, $g_n(x) = nx^n$. Show that g_n converges almost everywhere to zero. Is there a function h in $L^1([0,1])$ such that $|g_n(x)| \leq h(x)$ for almost all x in $[0,1]$?
(ii). If f is in $L^\infty([0,1])$ and f is continuous at 1, show that $\int_0^1 nx^n f(x)dx$ converges to $f(1)$.
Hint: set $x^{n+1} = y$.
(iii). If we only assume that f is in $L^1([0,1])$ and f is continuous at 1, does $\int_0^1 nx^n f(x)dx$ converges to $f(1)$?

Exercise 3:

Let X be a metric space and A and B two subsets of X such that $A \cap B = \emptyset$ and $A \cup B = X$.
Show that the following statements are equivalent:
- Any function $f : X \to \mathbb{R}$ is continuous if and only if the restriction of f to A and the restriction of f to B are continuous.
- A and B are both open and closed in X.