GCE: 503, Analysis and measure theory  
January 2020  

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Exercise 1:
Let $E$ be a measurable subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ a measurable function. For $a$ in $\mathbb{R}$, set $\omega_f(a) = |\{x \in E : f(x) > a\}|$ where $| \cdot |$ denotes the Lebesgue measure.

(i). If $f_k : E \to \mathbb{R}$ is a sequence of Lebesgue measurable, real-valued functions, such that $f_k \leq f_{k+1}$ and $f_k \to f$ almost everywhere, show that $\omega_{f_k} \leq \omega_{f_{k+1}}$ and $\omega_{f_k} \to \omega_f$.

(ii). Recall that $f_k$ converges in measure to $f$ if for all positive $\epsilon$, $|\{x \in E : |f_k(x) - f(x)| > \epsilon\}|$ tends to zero as $k$ tends to infinity.
If $f_k$ converges in measure to $f$ then show that $\limsup_{k \to \infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$, and $\liminf_{k \to \infty} \omega_{f_k}(a) \geq \omega_f(a + \epsilon)$, for every $\epsilon > 0$.

(iii). If $f_k$ converges in measure to $f$, show that $\omega_{f_k}(a) \to \omega_f(a)$ if $\omega_f$ is continuous at $a$.

Exercise 2:
Define the sequence of functions $g_n : [0, 1] \to \mathbb{R}$, $g_n(x) = nx^n$. Show that $g_n$ converges almost everywhere to zero. Is there a function $h$ in $L^1([0, 1])$ such that $|g_n(x)| \leq h(x)$ for almost all $x$ in $[0, 1]$?

(ii). If $f$ is in $L^\infty([0, 1])$ and $f$ is continuous at 1, show that $\int_0^1 nx^n f(x)dx$ converges to $f(1)$.
**Hint:** set $x_{n+1} = y$.

(iii). If we only assume that $f$ is in $L^1([0, 1])$ and $f$ is continuous at 1, does $\int_0^1 nx^n f(x)dx$ converges to $f(1)$?

Exercise 3:
Let $X$ be a metric space and $A$ and $B$ two subsets of $X$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Show that the following statements are equivalent:
- Any function $f : X \to \mathbb{R}$ is continuous if and only if the restriction of $f$ to $A$ and the restriction of $f$ to $B$ are continuous.
- $A$ and $B$ are both open and closed in $X$.  