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# 1:
Suppose that $S$ is a subset of $\mathbb{R}$. Let $F$ be the intersection of all closed sets that contain $S$. Show that $F$ is the set of all limits of sequences of points in $S$ that converge in $\mathbb{R}$. For the definition of closed, use only that a set is closed if and only if its complement is open.

# 2:
Let $f$ be a continuous and bounded real valued function on $\mathbb{R}$. Show that

$$
\|f\|_{\infty} = \sup \{|f(x)| : x \in \mathbb{R}\}.
$$

# 3:
Let $(K, \rho)$ be a compact metric space and $f : K \to K$ be such that

$$
\exists C \in (0,1), \quad \forall x, y \in K, \quad \rho(f(x), f(y)) \leq C \rho(x, y).
$$

(i). Let $x_1$ be in $K$ and set $x_{n+1} = f(x_n)$, for $n \geq 1$. Show that $x_n$ is a Cauchy sequence.

(ii). Show that $x_n$ converges to some $l$ in $K$ satisfying $l = f(l)$.

# 4:
Suppose $\{f_n\}$ is a bounded sequence in $L^1(\mathbb{R})$ and $f_n \to f$ a.e. Suppose also that $\forall M > 0, \exists n \in \mathbb{N}$ such that

$$
\int_{\mathbb{R} \setminus [-M, M]} |f_n| > 1.
$$

Show that $\|f_n\|_1 \not\to \|f\|_1$. 