Real Analysis GCE Name:__________________________

January, 2021

1:

Suppose that S is a subset of $\mathbb{R}$. Let F be the intersection of all closed sets that contain S. Show that F is the set of all limits of sequences of points in S that converge in $\mathbb{R}$. For the definition of closed, use only that a set is closed if and only if its complement is open.

2:

Lef f be a continuous and bounded real valued function on $\mathbb{R}$. Show that

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}.$$

3:

Let (K, ρ) be a compact metric space and $f : K \to K$ be such that

$$\exists C \in (0,1), \quad \forall x, y \in K, \quad \rho(f(x), f(y)) \le C\rho(x,y).$$

(i). Let x_1 be in K and set $x_{n+1} = f(x_n)$, for $n \ge 1$. Show that x_n is a Cauchy sequence.

(ii). Show that x_n converges to some l in K satisfying $l = f(l)$.

4:

Suppose $\{f_n\}$ is a bounded sequence in $L^1(\mathbb{R})$ and $f_n \to f$ a.e. Suppose also that $\forall M > 0$, $\exists n \in \mathbb{N}$ such that

$$\int_{\mathbb{R}\setminus[-M,M]} |f_n| > 1.$$

Show that $\|f_n\|_1 \not\to \|f\|_1$.