WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics - I
January, 2020

Note: The problems are arbitrarily ordered, not necessarily according to difficulty. Please show a clear logic of your solution. If you cannot solve a problem perfectly, still show your idea on solving the problem.

1. (20 points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2.
   (a) (10 points) Find the approximate distribution of $1/\bar{X}$ For a large sample. Is this valid for all values of μ?
   (b) (10 points) Show that $1/\bar{X}$ bias for $1/\mu$ but asymptotically unbiased.

2. (20 points) Assume existence
   (a) (8 points) State the rule of using the moment generation function (MGF) $M_X(t)$ to get any moment of X.
   (b) (12 points) Proof the above rule.

3. (20 points) Suppose that X and Z are independent normal random variables with mean 0 and variances σ_X^2 and σ_Z^2, respectively, i.e., $X \sim N(0, \sigma_X^2)$ and $Z \sim N(0, \sigma_Z^2)$. Let $Y = \beta_0 + \beta_1 X + Z$ where β_0 and β_1 are some constants.
   (a) (6 points) Let μ_Y and σ_Y^2 be the mean and variance of Y, and σ_{XY} be the covariance between X and Y. Express β_0, β_1, and σ_Z^2 in terms of μ_Y σ_Y^2, σ_X^2, and σ_{XY}.
   (b) (14 points) Suppose that $X_1, \ldots, X_n$ is a sequence of independent and identically distributed (iid) Normal random variables with mean 0 and variance σ_X^2, denoted by $X_1, \ldots, X_n \overset{iid}{\sim} N(0, \sigma_X^2)$, and that $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0, \sigma_Z^2)$ and Z_j is independent of X_k for any j and k. Let $Y_j = \beta_0 + \beta_1 X_j + Z_j$, $j = 1, \ldots, n$. Define

   $$b_{1,n} = \frac{\sum_{j=1}^n (X_j - \bar{X})(Y_j - \bar{Y})}{\sum_{j=1}^n (X_j - \bar{X})^2},$$

   where $\bar{X} = n^{-1} \sum_{j=1}^n X_j$ and $\bar{Y} = n^{-1} \sum_{j=1}^n Y_j$.
      (a) Prove that $b_{1,n}$ converges to β_1 in probability, as $n \to \infty$.
      (b) Find the conditional distribution of $b_{1,n}$ given $X_1 = x_1, \ldots, X_n = x_n$.

4. (20 points) Suppose X and Y are i.i.d. standard normal random variables. Let $Z = \min(X, Y)$. Show that $Z^2 \sim \chi_1^2$.

5. (20 points) Let $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$. Let $Y_i = X_{(i)}$ denote the i^{th} order statistic, $i = 1, \ldots, n$. Starting with the joint pdf of $Y_1, \ldots, Y_n$, derive the marginal pdf of $Y_r, r = 1, \ldots, n$. [Hint: The joint pdf of $Y_1, \ldots, Y_n$ is $f(y_1, \ldots, y_n) = n! \prod_{i=1}^n f(y_i), -\infty < y_1 <, \ldots, < y_n < \infty$.] If $X_1, \ldots, X_n \overset{iid}{\sim}$ Beta$(2, 1)$, deduce the pdf of $Y_r, r = 1, \ldots, n$. Is Y_r a beta random variable?

6. (20 points) Let $Y = (Y_1, \ldots, Y_n)$ have the joint pmf,

$$f(y_1, \ldots, y_n) = \frac{\beta^\alpha \Gamma(a+\alpha)}{b\Gamma(\alpha)(n+\beta)^{a+\alpha}}, y_i = 0, 1, 2, \ldots, \infty,$$

where $a = \sum_{i=1}^n y_i$ and $b = \prod_{i=1}^n y_i!$. Find $\text{Cor}(Y_i, Y_j), i, j = 1, \ldots, n$ [Hint: Find Z such that for $i = 1, \ldots, n$, $Y_i \mid Z$ are independent and identically distributed.]