WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 540 Probability and Mathematical Statistics - I
May, 2019

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly. Also, the problems are arbitrarily ordered, not necessarily according to difficulty.

1. (20 points) Suppose that X and Y are independent $\exp(\lambda)$ random variables with density

$$f(x) = \frac{1}{\lambda}\exp(-x/\lambda); \quad x > 0;$$

   (a) (10 points) Show that the sum $X + Y$ and the ratio X/Y are independent.

   (b) (10 points) Let $Z = \frac{X}{X+Y}$, show that for $0 < z < 1$, $F_Z(z) = P(Z \le z) = z$, i.e. the random variable Z is uniformly distributed over $(0, 1)$.

2. (20 points) Let $X \sim \text{Gamma}(\alpha, \beta), \alpha \ge 1$. Let $G(x)$ denote the cdf of X and let $b > 0$ be a constant. Show that $E(G(bX)) \le \frac{b}{b+1}\Gamma(2\alpha)$. What is the value of $E(G(bX))$ if $b = 1$?

3. (20 points) Let $f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}, -\infty < x, y < \infty$. Let

$$W = \frac{1}{(1-\rho^2)}(X^2 - 2\rho XY + Y^2).$$

   Show that $W \sim \chi^2_2$.

4. (20 points) Let $X_1, \ldots, X_n \mid \mu, \lambda \overset{ind}{\sim} f(x \mid \mu, \lambda)$, where

$$f(x \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}, x > 0.$$

   We may write $X_i \mid \mu, \lambda \overset{ind}{\sim} IG(\mu, \lambda), i = 1, \ldots, n$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Show that $\bar{X} \mid \mu, \lambda \sim IG(\mu, n\lambda)$.

5. (20 points) Consider a survey for studying whether Trump is favored in the 2020 presidential election. Questionnaires are to be sent to N independent randomly selected voters. Let $p \in (0, 1)$ be the proportion of all voters in the population in favor of Trump, and $\pi \in (0, 1)$ be the proportion of all voters who would return questionnaires. Assume whether or not returning questionnaire is irrelevant to whether or not Trump is favored, and all returned questionnaires tell the truth. What is the distribution for the number of returned questionnaires in favor of Trump?

   (a) (10 points) Let Y be the number of questionnaires to be returned, and X of them are in favor of Trump. Obtain the joint cumulative distribution function of (X, Y), $F(x, y)$, for any $x, y \in \mathbb{R}$.

(b) (10 points) Obtain the probability dense function of X, $f_X(x)$, for any $x \in \mathbb{R}$.

6. (20 points) Let $X_n \sim \text{Binomial}(n, p)$ where the positive integer n is large and $0 < p < 1$. Let $g(\theta) = \theta^3 - \theta$.

   (a) (10 points) Find the the limiting distribution of $\sqrt{n}\left[g\left(\frac{X_n}{n}\right) - c_1\right]$ for appropriate constant c_1 when $p = \frac{1}{3}$.

   (b) (10 points) Find the the limiting distribution of $n\left[g\left(\frac{X_n}{n}\right) - c_2\right]$ for appropriate constant c_2 when $p = \frac{1}{\sqrt{3}}$.