1. (20 points) Suppose that \(X_1, X_2, \cdots, X_n\) is a sample of i.i.d. observations drawn from a distribution function \(F\). The empirical distribution function is defined as

\[
\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq t), \quad \forall t \in (-\infty, \infty)
\]

where \(I(\cdot)\) is the indicator function.

(a) (6 points) Show that \(\hat{F}_n(t)\) is an unbiased estimator of \(F(t)\).

(b) (6 points) Specify the distribution of \(\hat{F}_n(t)\).

(c) (8 points) For any fixed \(t\), show that

\[
\sqrt{n}\{\hat{F}_n(t) - F(t)\} \xrightarrow{d} N(0, \nu(t))
\]

as \(n \to \infty\). Determine the value of asymptotic variance \(\nu(t)\). Here \(\xrightarrow{d}\) represents convergence in distribution.

2. (20 points) Let \(X_1, \ldots, X_n \mid \theta \sim \text{Normal}(\theta, \theta^2), \theta > 0\). Let \(T = \omega_1 \bar{X} + \omega_2 S^{-1}, \omega_1\) and \(S\) are respectively the sample mean and the sample variance. Find \(\omega_1\) and \(\omega_2\) so that \(T\) is the minimum variance unbiased estimator of \(\theta\). [Hint: \(E(S) = a\theta\), where \(a = \frac{\Gamma(n/2)\sqrt{2}}{\Gamma((n-1)/2)\sqrt{n}}, n \geq 2, a > 1\).]

3. (20 points) Let \(X_1, \ldots, X_n\) are iid from Bernoulli(p) where \(n \geq 2\) and \(0 < p < 1\) is the known parameter.

(a) (10 points) Derive the uniformly minimum-variance unbiased estimator (UMVUE) of \(\tau(p)\), where \(\tau(p) = e^2(p(1-p))\).

(b) (10 points) Find the Cramér–Rao lower bound for estimating \(\tau(p) = e^2(p(1-p))\).

4. (20 points) Suppose that \(X_1, \ldots, X_n\) are iid samples from a common discrete distribution

\[
P(X_k = x) = \frac{\exp(\theta x)}{\exp(-\theta) + 1 + \exp(\theta)}, \quad x = -1, 0, 1,
\]

for \(k = 1, \ldots, n\), where \(\theta\) is a real-valued unknown parameter.

(a) (10 points) Find a minimal sufficient statistic for \(\theta\) based on \((X_1, \ldots, X_n)\).

(b) (10 points) Find the maximum likelihood estimator of \(\theta\) based on \((X_1, \ldots, X_n)\).
5. (20 points) Let $X_1, \ldots, X_n$ denote the order statistics of a sample $X_1, \ldots, X_n | \theta \overset{\text{iid}}{\sim} \text{Uniform}(0, \theta)$. Let $R = X_{(n)} - X_{(1)}$ denote the sample range. In the textbook by Casella and Berger, the pdf of $R$ is

$$f(r) = \frac{n(n-1)r^{n-2}(\theta - r)}{\theta^n}, 0 < r < \theta.$$ 

Show that, using the pivotal method, a $100(1 - \alpha)\%$ equal-tailed confidence interval for $\theta$ is $(b^{-1}R, a^{-1}R)$, where

$$[n - (n-1)a]a^{n-1} = \alpha/2 \quad \text{and} \quad [n - (n-1)b]b^{n-1} = 1 - \alpha/2.$$ 

6. (20 points) Let $X_1, X_2, \ldots, X_n$ be a sample of size $n$ with common probability density function $f(x|\theta)$. Denote $\hat{\theta}$ the maximum likelihood estimator of $\theta$. Let $g(\cdot)$ be a continuous function. It is well known that if $f(x|\theta)$ is “well-behaved” (e.g., under some regularity conditions on $f(x|\theta)$), $g(\hat{\theta})$ is a consistent and asymptotically efficient estimator of $g(\theta)$.

(a) (6 points) Use statistical notations to express the statement that “$g(\hat{\theta})$ is a consistent and asymptotically efficient estimator of $g(\theta)$”.

(b) (14 points) Give an outline of proof for the above statement. You can focus on the key ideas and steps, ignoring secondary details for the regularity conditions on $f(x|\theta)$. [Hint: You can consider applying Taylor expansion to the first derivative of the log likelihood function.]