WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics - II
August, 2018

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. Let $X_1, \ldots, X_n | \beta \ iid \sim \text{Gamma}(\alpha, \beta)$, where $\alpha$ is assumed known. Let $R = S/\bar{X}$, where $\bar{X}$ and $S$ are respectively the sample mean and sample standard deviation. What parameter does $R$ estimate? Is $R$ an unbiased estimator? Argue that $R$ is stochastically independent of $\bar{X}$. Find $E(R^2)$ and give an approximation to $E(R)$. [For $X \sim \text{Gamma}(a, b)$, $E(X) = a/b$, $\text{Var}(X) = a/b^2$.]

2. Let $X_1, \ldots, X_n | \theta$ be independent and identically distributed with

$$f(x | \theta) = \frac{2x}{\theta^2}, 0 < x < \theta.$$  

Find the MLE, $\hat{\theta}$, of $\theta$ and its asymptotic distribution (as $n$ goes to infinity). What optimality property does $\hat{\theta}$ have?

3. A forest has $N$ (unknown) monkeys. A random sample of $n$ monkeys is selected from the forest, tagged and released back into the forest. After a few days, a random sample of $m$ monkeys is selected and $Y$ monkeys are found with the tags. Find an estimator of $N$. Approximate the mean and variance of your estimator. In this experiment, with $n = 25$ and $m = 16$, $Y$ was observed to be 8. Find an approximate 95% confidence interval for $N$.

4. Let $Y_1, \ldots, Y_n$ be iid from a one parameter exponential family with pdf or pmf $f(y|\theta)$ with complete sufficient statistic $T(Y) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$ and $X$ has a known distribution with known $E(X)$ and known variance $V(X)$. Let $W_n = cT(Y)$ be an estimator of $\theta$ where $c$ is a constant.

(a) Find the mean square error (MSE) of $W_n$ as a function of $c$ (and of $n$, $E(X)$ and $V(X)$).
(b) Find the value of $c$ that minimizes the MSE. Prove that your value is the minimizer.
(c) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.

5. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with mean $\theta$.

(a) Show that $T = \sum_{i=1}^n X_i$ is complete sufficient statistic for $\theta$.
(b) For $a > 0$, find the uniformly minimum variance unbiased estimator (UMVUE) of $g(\theta) = e^{a\theta}$.
(c) Prove the identity:

\[ E[2^{X_i}|T] = \left(1 + \frac{1}{n}\right)^T. \]

6. Let \( X_1, \ldots, X_n \overset{iid}{\sim} \Gamma(1, \theta) \) and independently \( Y_1, \ldots, Y_m \overset{iid}{\sim} \Gamma(1, \mu) \). Consider testing \( H_o : \theta = \mu \) versus \( H_1 : \theta \neq \mu \).

   (a) Find the likelihood ratio test (LRT).
   (b) Let \( T = \frac{\bar{X}}{n \bar{X} + m \bar{Y}} \). Show that the LRT can be based on \( T \).
   (c) Find the distribution of \( T \) when \( H_o \) is true.