WPI Mathematical Sciences Ph.D. General Comprehensive Exam
MA 541 Probability and Mathematical Statistics II

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly. Also, the problems are arbitrarily ordered, not necessarily according to difficulty.

1. (20 points) Let $X_1, ..., X_n$ be i.i.d. with the marginal probability mass function $f(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$, $x = 0, 1, 2, ...$ Find the asymptotic variance of the MLE for $e^{-\lambda}$.

2. (20 points) Let $\mathbf{X} = (X_1, ..., X_n)$ be a random sample, with either continuous or discrete distribution. Let $W(\mathbf{X})$ be a test statistic for testing hypotheses $H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \notin \Theta_0$ such that large values of W support that H_1 is true. For each sample point $\mathbf{x}$, define
$$p(\mathbf{x}) = \sup_{\theta \in \Theta_0} P_\theta(W(\mathbf{X}) \geq W(\mathbf{x})).$$
Show that $p(\mathbf{X})$ is a valid p-value.

3. (20 points) Let $X_{i1}, \ldots, X_{in_i} \overset{ind}{\sim} \text{Normal}(\mu_i, \sigma^2), i = 1, \ldots, \ell$, where σ^2 and $\mu_1, \ldots, \mu_n$ are unknown. Find the MLE of σ^2 and find an unbiased estimator of σ^2.

4. (20 points) Let $X_1, \ldots, X_n$ be a random sample such that $X_i = \ln(Y_i) \sim \text{Normal}(\mu, \sigma^2), i = 1, \ldots, n$. Let $Y_{(1)}, \ldots, Y_{(n)}$ denote the order statistics. Let $\hat{\theta} = \frac{\sum_{i=1}^n Y_{(i)}/n}{\{\prod_{i=1}^n Y_{(i)}\}^{1/n}}$. Find θ such that $\hat{\theta}$ is a consistent estimator of θ. What is the asymptotic distribution of $\sqrt{n}\hat{\theta}$?

5. (20 points) Let $X_1, \ldots, X_n$ be a random sample from a population with mean, μ, and variance, σ^2. Find $\text{Cor}(X_i - \bar{X}, X_j - \bar{X}), i \neq j$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ is the sample mean.

6. (20 points) Let $X_1, ..., X_n$ be a random sample from a population with pdf
$$f(x/\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \;:\; \theta > 0, \\ 0 & \textit{elsewhere}. \end{cases}$$
   (a) (10 points) Let $W_i = -\ln(X_i), i = 1, 2, \cdots, n$. Show that $2\theta \sum_{i=1}^n W_i$ has a chi-square distribution with $2n$ degrees of freedom.
   (b) (10 points) Find a minimum variance unbiased estimator (MVUE) for θ.