# WPI Mathematical Sciences Ph.D. General Comprehensive Exam
## MA 541 Probability and Mathematical Statistics II
## May, 2019

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly. Also, the problems are arbitrarily ordered, not necessarily according to difficulty.

1. (20 points) Let $U$ be an unbiased estimator of $\theta$ and $R$ a biased estimator of $\theta$, where $\mathrm{Var}(U) = \sigma_1^2, \mathrm{Var}(R) = \sigma_2^2$ and $\mathrm{Cor}(U, R) = \rho \geq 0$. Let $S = \lambda U + (1-\lambda)R, 0 \leq \lambda \leq 1$. Show that $S$ is less biased than $R$ and $\mathrm{Var}(S)$ is minimized if $\rho \leq \min(\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1})$.

2. (20 points) Let $X_1, \ldots, X_n \mid \mu, \lambda \overset{ind}{\sim} f(x \mid \mu, \lambda)$, where

$$f(x \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}, x > 0.$$

   Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{n}{\sum_{i=1}^n (\frac{1}{X_i} - \frac{1}{\bar{X}})}$. Show that $S^2$ is actually positive. Then show that $\bar{X}$ and $S^2$ are independent. [Hint: $\frac{n\lambda}{S^2} \sim \chi^2_{n-1}$.]

3. (20 points) Let $X_1, \ldots, X_n \mid \theta \overset{ind}{\sim} \mathrm{Uniform}(0, \theta)$ and let $\bar{X}$ and $X_{(n)}$ denote respectively the sample mean and the largest order statistic. Use a simple argument to show that $\frac{\mathrm{Var}(\bar{X})}{\mathrm{Var}(X_{(n)})} \geq \left(\frac{n+1}{2n}\right)^2$. Then, prove the inequality directly.

4. (20 points) Let $X_1, \cdots, X_n$ be a random sample from a distribution with the pdf given by

$$f(x|\theta) = \theta^{-c} c x^{c-1} e{-}(x/\theta)^c I(x > 0),$$

   where $c > 0$ is a known constant.

   (a) (10 points) Find the maximum likelihood estimator (MLE) of $\theta$.

   (b) (10 points) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.

5. (20 points) In general, the log likelihood ratio test statistic is defined as

$$T(\mathbf{x}) = -2 \log \frac{\sup_{\theta \in \Theta_0} L(\theta|\mathbf{x})}{\sup_{\theta \in \Theta} L(\theta|\mathbf{x})},$$

   where $\mathbf{x} = (x_1, \ldots, x_n)$ is an observed sample, $L(\theta|\mathbf{x})$ is the likelihood function, $\Theta$ is the parameter space of $\theta$, and $\Theta_0$ is the null space under $H_0$. Consider the two-sided simple test:

$$H_0 : \theta \in \Theta_0 = \{\theta_0\} \quad \text{versus} \quad H_1 : \theta \notin \Theta_0.$$

Show the convergence in distribution:

$$T(\mathbf{X}) \xrightarrow{D} \chi_1^2.$$

[Hint: The MLE is asymptotically efficient. You can also assume the distribution of $\mathbf{X}$ is "well behaved", i.e., it satisfies regularity conditions whenever needed.]

6. (20 points) Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\lambda\sqrt{2\pi}}\frac{1}{x} exp\left[\frac{-(\log(x))^2}{2\lambda^2}\right],$$

where $\lambda > 0$ and $0 \le x \le 1$.

   (a) (10 points) What is the UMP (uniformly most powerful) level $\alpha$ test for $H_0 : \lambda = 1$ vs $H_1 : \lambda = 2$?

   (b) (10 points) If possible, find the UMP level $\alpha$ test for $H_0 : \lambda = 1$ vs $H_1 : \lambda > 1$