

# WORCESTER POLYTECHNIC INSTITUTE

## 32nd INVITATIONAL MATH MEET

October 15, 2019

### INDIVIDUAL EXAM with Answers

**DIRECTIONS:** Please write your answers on the Individual Answer Sheet provided. This part of the contest is 45 minutes long. Questions 1-4 are each worth 1 point. Questions 5-8 are each worth 2 points. Questions 9-11 are each worth 3 points. Calculators and other electronics **MAY NOT** be used.

#### 1 Point Each:

1. What are all the nonnegative integers  $x$  for which  $(x-6)(x+14)$  is a perfect square?

**ans:** 6, 22

2. A cow on 30' rope is hooked to a corner of a 10' x 20' barn.

How many square feet of grazing area does she have? **ans:**  $800\pi$  ft<sup>2</sup>

3. Assume that all of the following are  $n \times n$  invertible matrices:  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{X}$ . Solve the following equation for  $\mathbf{X}$ :  $\mathbf{A}^2 \mathbf{B}^{-1} \mathbf{X} \mathbf{C} + \mathbf{F} = \mathbf{I}_n$ . ( $\mathbf{I}_n$  denotes the  $n \times n$  identity matrix)

**ans:**  $\mathbf{B} \mathbf{A}^{-2} (\mathbf{I}_n - \mathbf{F}) \mathbf{C}^{-1}$

4. Consider the circle  $x^2 + y^2 = 16$  and the ellipse  $x^2/16 + y^2/9 = 1$ . What is the area inside the circle and outside the ellipse?

**ans:**  $4\pi$

**2 Points Each:**

5. If  $x$  is a real number such that  $2x^3 + 4x^2 + 6x + 8 = 2468$ , what is the value of  $x^3 + 9x^2 + 8x + 8$  ??  
**ans:** 1988

6. What is the area of the smallest circle centered at the origin which touches both branches of the hyperbola  $xy = 2$  ?  
**ans:**  $4\pi$

7. Please evaluate

$$\sum_{j=2}^5 \sum_{i=1}^6 (ji^2 + 2)$$

**ans:** 1322

8. In a certain school, 60% of the students have a dog at home. Suppose that 8 students are sampled. What is the probability that exactly 5 have a dog at home? (no need to simplify your resulting answer)

**ans:**  $(8!/(5!3!)) * (0.6)^5 (0.4)^3$

**3 Points Each:**

9. Consider the infinite series  $1 + \frac{i}{2} - \frac{1}{4} - \frac{i}{8} + \frac{1}{16} + \frac{i}{32} \dots$  where  $i = \sqrt{-1}$  which converges to a point in the complex plane. If the distance of that point from the origin is  $\frac{a\sqrt{5}}{b}$  find  $a$  and  $b$  . **ans:**  $a=2$   $b = 5$

10. What is the value of  $k$  for which

$$\frac{2000!}{1000!} = k(1 \times 3 \times 5 \times 7 \times \dots \times 1997 \times 1999) \quad ?$$

**ans:**  $k = 2^{1000}$

11. How many **0**'s are at the end of **62!**

**ans:** 14