Problem 1: Let \( P_k(x) \) denote the vector space of polynomials with real coefficients of degree \( k \) or less in \( x \). Consider the linear transformation \( T : P_3(x) \rightarrow P_1(x) \) given by second differentiation, i.e., by \( T(p) = p'' \in P_1(x) \) for \( p \in P_3(x) \).

Find the matrix representation of \( T \) with respect to the bases \( \{1 + x, 1 - x, x + x^2, x^2 - x^3\} \) for \( P_3(x) \) and \( \{1, x\} \) for \( P_1(x) \).

Problem 2: Let \( A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & -4 & -4 \end{pmatrix} \).

(a) What is the rank of \( A \)?
(b) What is the determinant of \( A \)?
(c) Find the eigenvalues and eigenvectors of \( A \).
(d) Find the characteristic polynomial of \( A \).
(e) Find the transformation matrix \( M \) and its inverse such that \( J = M^{-1}AM \) is the Jordan canonical form of \( A \).
(f) Does it make a difference if you do your computations over the real numbers or over the complex numbers? Justify your answer.

Problem 3: This problem involves the matrix exponential \( \exp(M) \) for a square matrix \( M \).

(a) Compute \( \exp(At) \) if \( A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \).
(b) Prove that, if \( AB = BA \), then \( \exp(A) \exp(B) = \exp(A + B) \).
(c) Prove that, if \( A \) is skew-symmetric (i.e., \( A^\top = -A \)) then \( \exp(A) \) is an orthogonal matrix.

Problem 4: Let \( A \) be an \( n \times n \) complex Hermitian matrix with largest eigenvalue \( \lambda_1 \). Let \( B \) be the \( (n-1) \times (n-1) \) matrix obtained by deleting
the first row and first column of $A$. If $\mu_1$ is the largest eigenvalue of $B$, prove that $\mu_1 \leq \lambda_1$.

**Problem 5:** Suppose that $T$ is an $n \times n$ linear transformation over the field $\mathbb{Q}$ of rational numbers satisfying $T^2 = T^{-1} - T$. Prove that $n \equiv 0 \mod 3$.

**Problem 6:** Let $V = C^\infty([0, 1])$ be the real inner product space of infinitely differentiable functions on the interval $[0, 1]$ with inner product

$$\langle f, g \rangle := \int_0^1 f(t)g(t) \, dt .$$

The differential operator $T = \frac{d}{dt}$ is a linear operator on $V$. The Riesz Representation Theorem guarantees the existence and uniqueness of the adjoint operator $T^*$ of $T$. Give the meaning of $T^*$ and in the special case where $f(0) = f(1) = 0$, find a simple expression for the function $T^*f$. 