Connecting Calculus to Linear Programming

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Motivation

Goal: To help students make connections between high school math and real world applications of mathematics.

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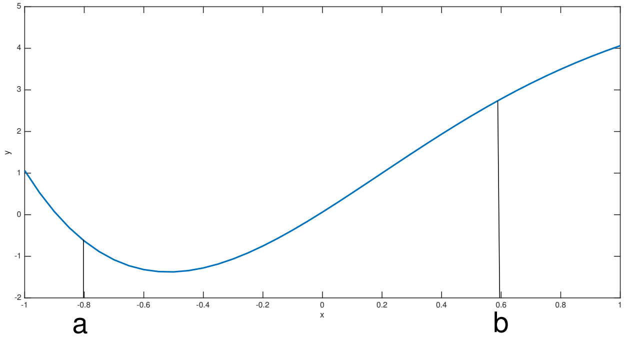
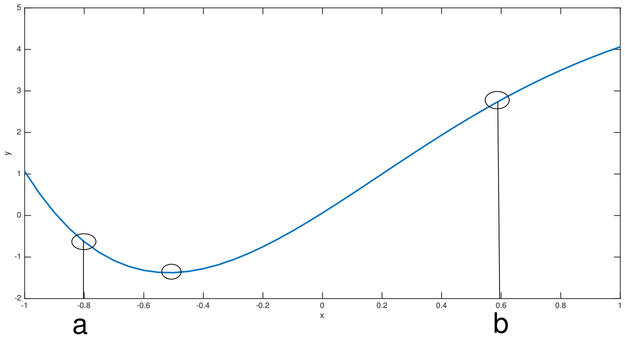
- Linear programming is based on a simple idea from calculus.
- We can connect calculus to real-world industrial problems.
- We'll explore the some basic principles of linear programming and some modern-day applications.

Continuous Functions on Closed Intervals



What can we say about a continuous function on a closed interval?

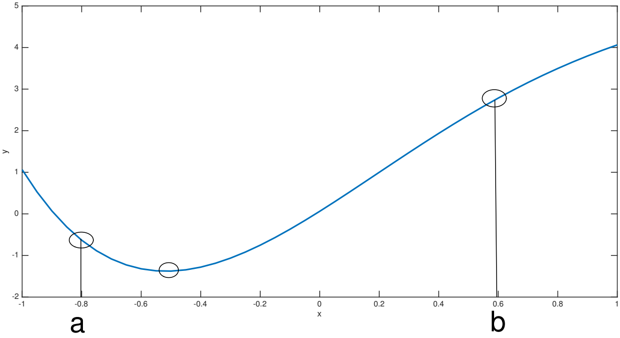
Continuous Functions on Closed Intervals



Theorem

If f is continuous on $[a, b]$ then f attains both an absolute minimum value and an absolute maximum value on $[a, b]$.

Continuous Functions on Closed Intervals



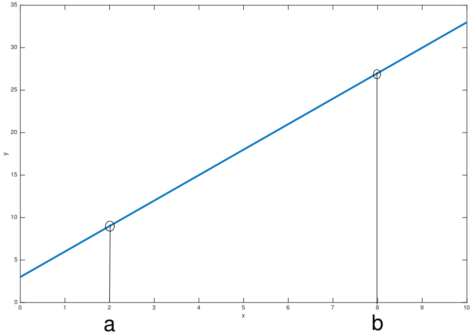
Theorem

If f is continuous on $[a, b]$ then f attains both an absolute minimum value and an absolute maximum value on $[a, b]$. Further these occur at critical points of f or endpoints of $[a, b]$.

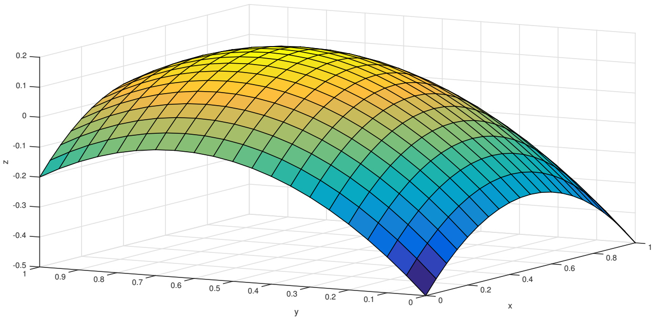
Linear Case

Theorem

If f is ~~continuous~~ linear on $[a, b]$ then f attains both an absolute minimum value and an absolute maximum value on $[a, b]$, and these occur at endpoints of $[a, b]$.

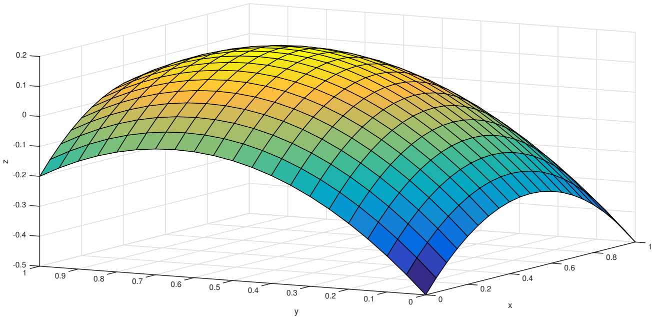


Higher-Dimensional Optimization



What does this mean for a continuous function $f(x, y)$ on a closed and bounded plane region R?

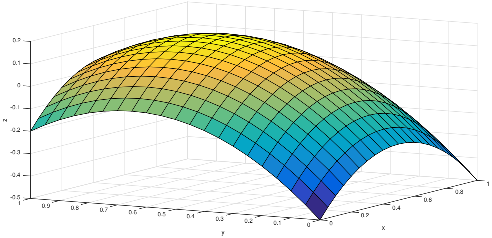
Higher-Dimensional Optimization



Theorem

If $f(x, y)$ is continuous on closed and bounded plane region R then f attains both an absolute minimum value and an absolute maximum value on R. Further, these values occur either at critical points of f in the interior of R or on the boundary of R.

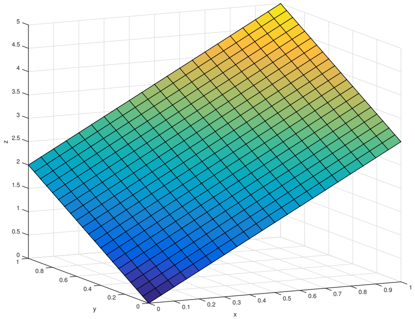
Higher-Dimensional Optimization



Theorem

If $f(x_1, x_2, \ldots, x_n)$ is continuous on closed and bounded region $R \subset \mathbb{R}^n$ then f attains both an absolute minimum value and an absolute maximum value on R. Further, these values occur either at critical points of f in the interior of R or on the boundary of R.

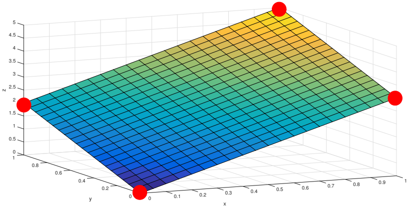
Higher-Dimensional Linearity



Theorem

If $f(x_1, x_2, \ldots, x_n)$ is linear on closed and bounded region $R \subset \mathbb{R}^n$ defined by a system of linear constraints then f attains both an absolute minimum value and an absolute maximum value on R. Further, these values occur on the boundary of R.

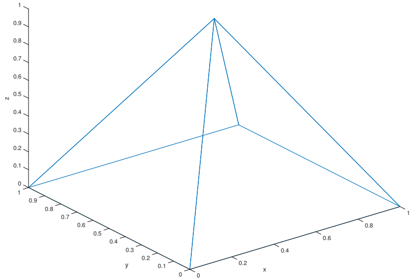
Higher-Dimensional Linearity



Theorem

If $f(x_1, x_2, \ldots, x_n)$ is linear on closed and bounded region $R \subset \mathbb{R}^n$ defined by a system of linear constraints then f attains both an absolute minimum value and an absolute maximum value on R. Further, these values occur on corner points of the boundary of R.

Higher-Dimensional Linearity



Example: A possible feasible region $f(x, y, z) = ax + by + cz$ subjuect to 5 linear constraints.

A solution to minimize f over this region will occur at a corner.

Linear Programming

- The Simplex Method was developed by George Dantzig in 1947.
- “programming” synonymous with “optimization”.
- Algorithm to traverse the corner points of the feasible polyhedron for a linear programming problem to find an optimal feasible solution.

Standard form for a linear programming problem:

$$\min \mathbf{x}^T \mathbf{c} \text{ such that } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}. \tag{1}$$

$\mathbf{x}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$ - n variables and m constraints.

Simplex Method

$\min \mathbf{x}^T\mathbf{c}$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

- For each constraint (row i of A), add a new **slack variable** y_i .
- New system: $\min \mathbf{x}^T\mathbf{c}$ such that $[A + I]\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$.
- Underdetermined (more variables than equations). $A + I$
- The slack now in the system is the key.
- Each slack variable is associated with a constraint boundary

Simplex Method

Simplex algorithm:

- Split entries of $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ into 2 subsets:
  - Basic variables: $\mathbf{x}_B \in \mathbb{R}^m$ (non-zero valued)
  - Non-basic variables: $\mathbf{x}_N \in \mathbb{R}^n$ (zero valued)
  - Represents a corner point of the feasible region.
- If not optimal, move to an adjacent corner point by swapping one entry in $\mathbf{x}_B$ with one entry in $\mathbf{x}_N$ and re-solving the system of equations.

Simplex Geometry

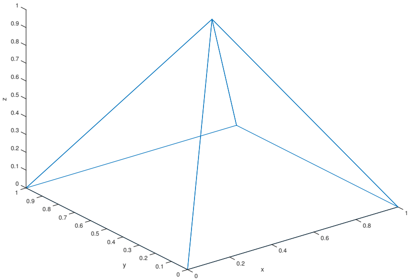
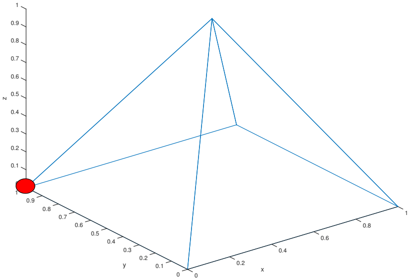


Figure: Feasible Region

Example: $\min f(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$ subject to 5 linear constraints.

Simplex Geometry



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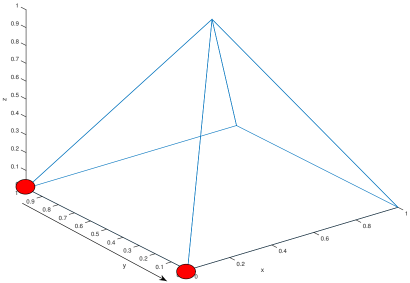
Choose initial basis to correspond to the origin.

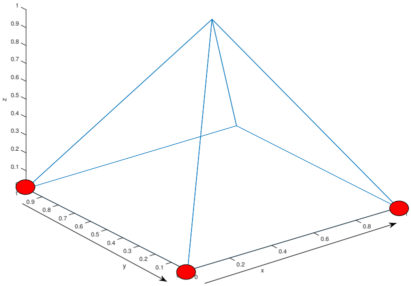
Simplex Geometry



Example: $\min f(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$ subject to 5 linear constraints.

If the objective function is not optimal, move to a better adjacent corner.

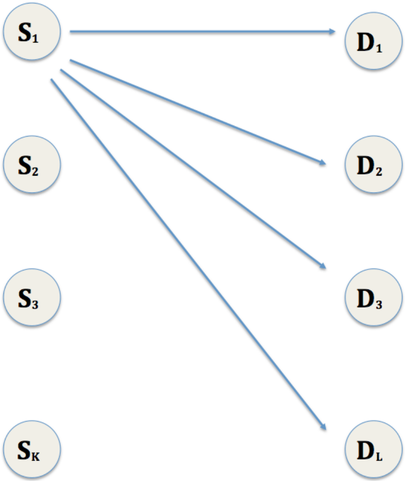
Simplex Geometry



Example: min $f(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$ subject to 5 linear constraints.

Repeat until the objective function is minimized (no remaining adjacent corners will reduce it).

Application: Shipping Network



Shipping Problem

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- Problem: Design a shipping schedule that satisfies demand at all retail locations at a minimal cost.

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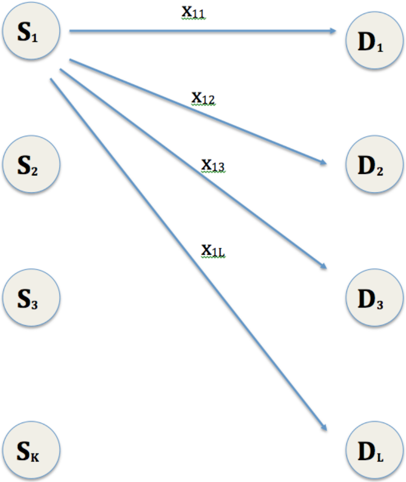
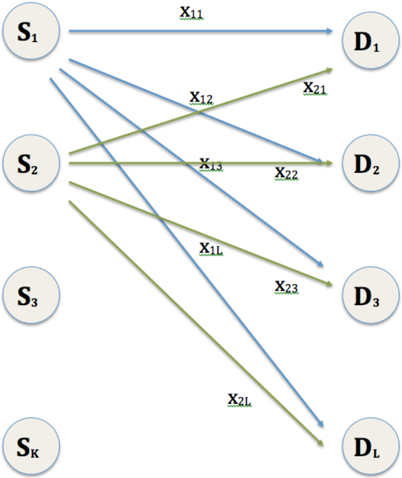
- Example: As of July 2017, Target Corp. has 37 distribution centers and $1,802$ stores.

Shipping Network



Shipping Network



Shipping Problem Mathematical Model

Constraints

- $x_{ij} \geq 0$ - amount of product shipped from S_i to D_j

Objective

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- Total cost of shipping schedule: $\sum_{i=1}^{K} \sum_{j=1}^{L} c_{ij} x_{ij}$

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Objective

- Total cost of shipping schedule: $\sum_{i=1}^{K} \sum_{j=1}^{L} c_{ij} x_{ij}$
- Minimize cost of shipping such that demand is satisfied
- Target example: $37 \times 1,802 = 66,674$ variables

Shipping Problem Mathematical Model

- Amount leaving S_i : $x_{i1} + x_{i2} + \cdots + x_{iL} \leq s_i$

$$A = \begin{bmatrix} 1 & \ldots & 1 & & & & & & \\ & & & 1 & \ldots & 1 & \ddots & & \\ & & & & & & 1 & \ldots & 1 \\ & \mathrm{I} & & & \mathrm{I} & & & \mathrm{I} & \end{bmatrix}$$

$$\min \sum_{i=1}^{K} \sum_{j=1}^{L} c_{ij} x_{ij} \text{ such that } A\mathbf{x} \begin{matrix} \leq \\ = \end{matrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{d} \end{bmatrix}, \mathbf{x} \geq \mathbf{0}.$$

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$$A = \begin{bmatrix} 1 \ \dots \ 1 & & & \\ & 1 \ \dots \ 1 & \ddots & \\ & & & 1 \ \dots \ 1 \\ \mathrm{I} & \mathrm{I} & & \mathrm{I} \end{bmatrix}$$

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TV Advertising Project Solution - Math 3231
Marcel Blais, A Term 2015

You have been hired as a consultant to determine a one-week television advertising campaign schedule for a company. This means you will recommend how much advertising time the company should buy for each television show. You have been asked to use Nielsen Media Research ratings data to find an optimal way to spend an advertising budget of $10,000,000. The higher the Nielsen rating, the larger the audience reached by the ad.

Using the Nielsen data and the price[1] per half-minute[2] of advertisement slots on the listed television shows, formulate your advertising scheduling problem as a linear programming problem.

Your client has also asked that you honor the following constraints:

TV Project - MA 3231

Your client has also asked that you honor the following constraints:

- There should be at least 4 minutes but no more than 9 minutes of total advertising on each day (except Saturday).
- Each of the major 4 networks, ABC, CBS, NBC, and FOX, should have at least 5 total minutes of ads for the week.
- You cannot have more than 1.5 minutes of ads in any single TV show.

1. Formulate this as a linear programming problem. Hand in a written formulation and explanation of your model.
2. Use a MS Excel file called *TV_Advertising.xlsx* to implement and solve your model. Upload your file to the course website.
3. Give the optimal advertising schedule. What conclusions can you make about how the advertising spots are priced?
4. Formulate the dual of this problem. How can we interpret it?

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[1] See *MA_3231_TV_Project_Data.xlsx*.
[2] Note that it is an industry standard for television ad pricing to be done for half-minutes.

TV Project - MA 3231

Mathematical model

- N: number of TV shows.
- x_i: number of $\frac{1}{2}$−minutes of advertising for television show i
- c_i: cost of $\frac{1}{2}$−minute of advertising for show i.
- r_i: Nielsen rating for show i.
- Objective: Maximize exposure weighted by Nielsen rating

$$\max \sum_{i=1}^{N} r_i x_i$$

- Constraints
  - Time per day
    $8 \leq \sum_{i \in \text{Day}} x_i \leq 18$ for Day $\in$ {M,T,W, Th, F}
  - Time per network
    $\sum_{i \in \text{Network}} x_i \geq 10$ for Network $\in$ {ABC, CBS, NBC, FOX}
  - Time per show
    $0 \leq x_i \leq 3$ for $i = 1, 2, \ldots, N$
  - Budget constraint
    $\sum^{N} c_i x_i \leq \$10,000,000$

References

1. Blais, Marcel., *MA 3231 Linear Programming Lecture Notes*, WPI Department of Mathematical Sciecnes, 2016.

2. Griva, Igor, Stephen G. Nash, and Ariela Sofer., *Linear and Nonlinear Optimization, Second Edition*. SIAM, 2009.

3. Hillier, Frederick and Gerald Lieberman., *Introduction to Operations Research, Tenth Edition*. McGraw Hill Education, 2015.