WPI Mathematical Sciences Ph.D. General Comprehensive Exam

MA 541 Probability and Mathematical Statistics-II

January, 2018

Note: Please make sure to write down your thinking process in bullet points even if you cannot solve the problems perfectly.

1. Let $Y_1, \ldots, Y_n \mid p \overset{iid}{\sim} \text{Bernoulli}(p), p \sim \text{Beta}(\alpha, \beta)$. Let $\hat{p}$ denote the Bayes estimator of p under squared error loss and let $M(p)$ denote the mean squared error of $\hat{p}$. Show that $M(p)$ is bounded in p provided $\alpha = \beta = \frac{1}{2}\sqrt{n}$. Give the bounded mean squared error.

2. A random sample of n adults is taken from a large population to estimate the proportion, π, of adults responding 'yes' to item, A, and each adult is asked to respond truthfully. A known proportion $0 < \gamma < 1$ of the adults answers A only, and the remaining proportion $1-\gamma$ does the following. Each respondent is asked to toss a biased coin (probability of heads is p, known). If the coin comes up heads, the respondent is asked to answer A; if it comes up tails, the respondent is asked to answer the opposite of A. Suppose there are y 'yeses' among the n adults. Find the maximum likelihood estimator of π and its standard error. Briefly discuss the quality of the maximum likelihood estimator.

3. Let $X_1, \ldots, X_n \mid \theta \overset{iid}{\sim} \text{Uniform}(0, \theta)$ and let $X_{(1)} = \min(X_1, \ldots, X_n)$ and $X_{(n)} = \max(X_1, \ldots, X_n)$. Find $f(x_{(1)} \mid x_{(n)})$. Are you surprised with your answer? Next, find $E(X_{(1)})$ and $E(X_{(1)} \mid X_{(n)})$. Deduce the important optimality result.

4. Let $Y_1, \cdots, Y_n$ be independent samples from the distribution with pdf containing the unknown parameter $\theta > 0$:
$$f_\theta(y) := \begin{cases} \frac{1}{\theta} y^{\frac{1}{\theta}-1} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$
   (a) Determine the method of moments estimator for θ using the Y_i.
   (b) Determine the maximum likelihood estimator for θ using the Y_i.
   (c) Ideally, estimator with a smaller variation should be better. Derive the asymptotic variances for the above two estimators.

5. Assume a random sample $\boldsymbol{X} = (X_1, \ldots, X_n)$ is drawn from a population given by a pdf $f(\cdot; \theta)$, $\theta \in \Theta$. Assume that for each such sample the likelihood function is maximized by exactly one point in Θ. Prove that a likelihood estimator $\hat{\theta}$ is always invariant to changes in the labels of $X_1, \ldots, X_n$.

6. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed (iid) random variables with probability density function

$$f(x) = \frac{2}{\sqrt{2\pi}\lambda} e^x exp\left(\frac{-(e^x - 1)^2}{2\lambda^2}\right),$$

where $x > 0$ and $\lambda > 0$.

(a) What is the UMP (uniformly most powerful) level α test for $H_0 : \lambda = 1$ vs. $H_1 : \lambda = 2$?

(b) If possible, find the UMP level α test for $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$.